Triangles, Squares or Hexagons?

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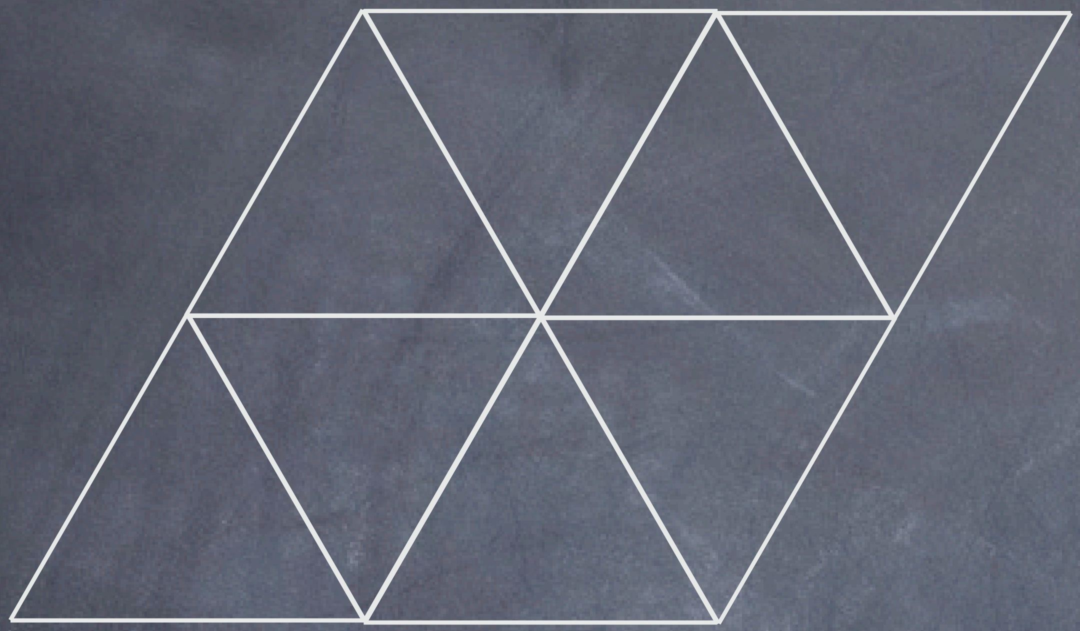
http://public.lanl.gov/ringler/ringler.html

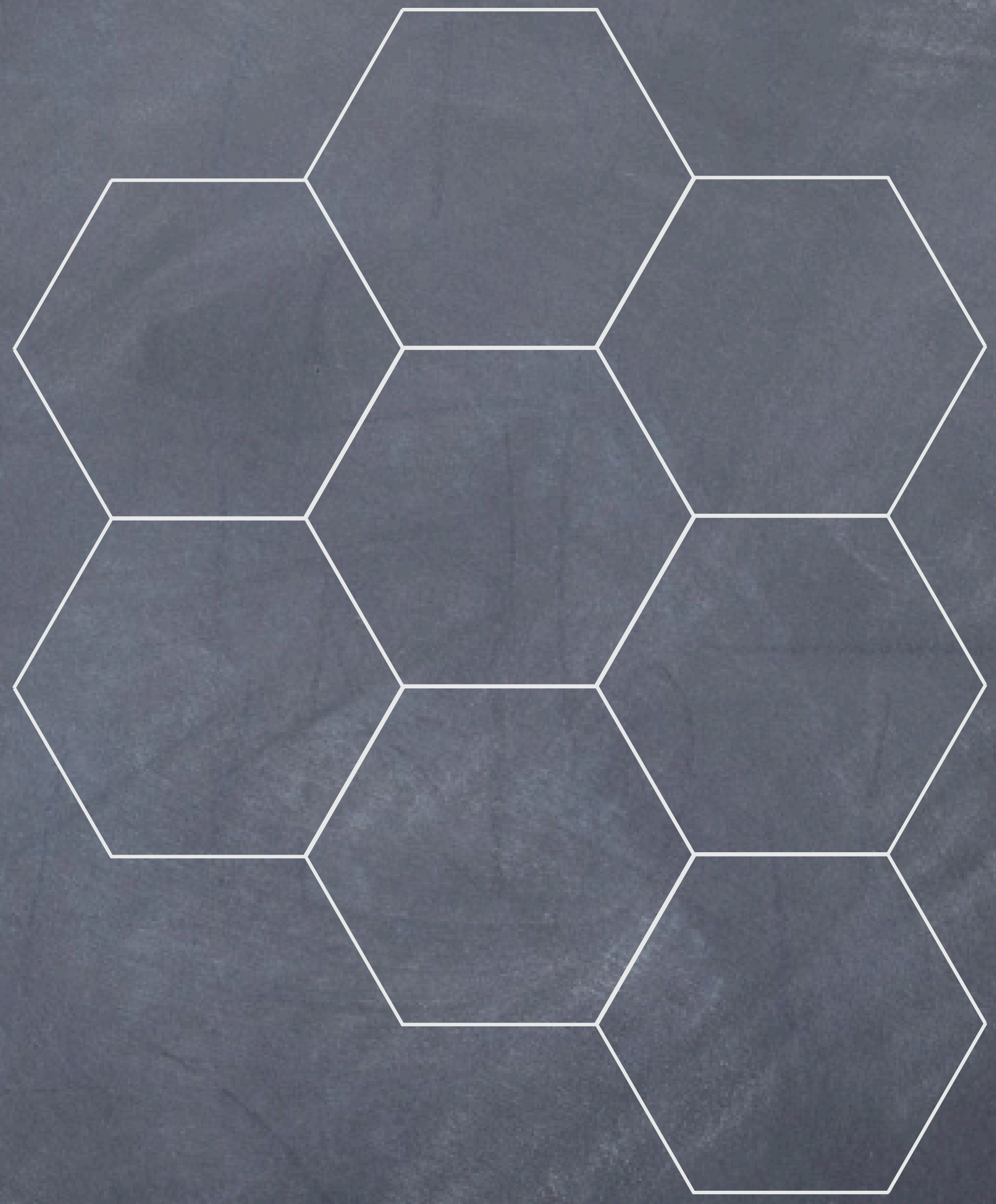
Purpose and Motivations

We have access to a number of gridding systems (triangles, squares and hexagons).

Is there any reason to choose one above the others or is the choice mostly arbitrary?

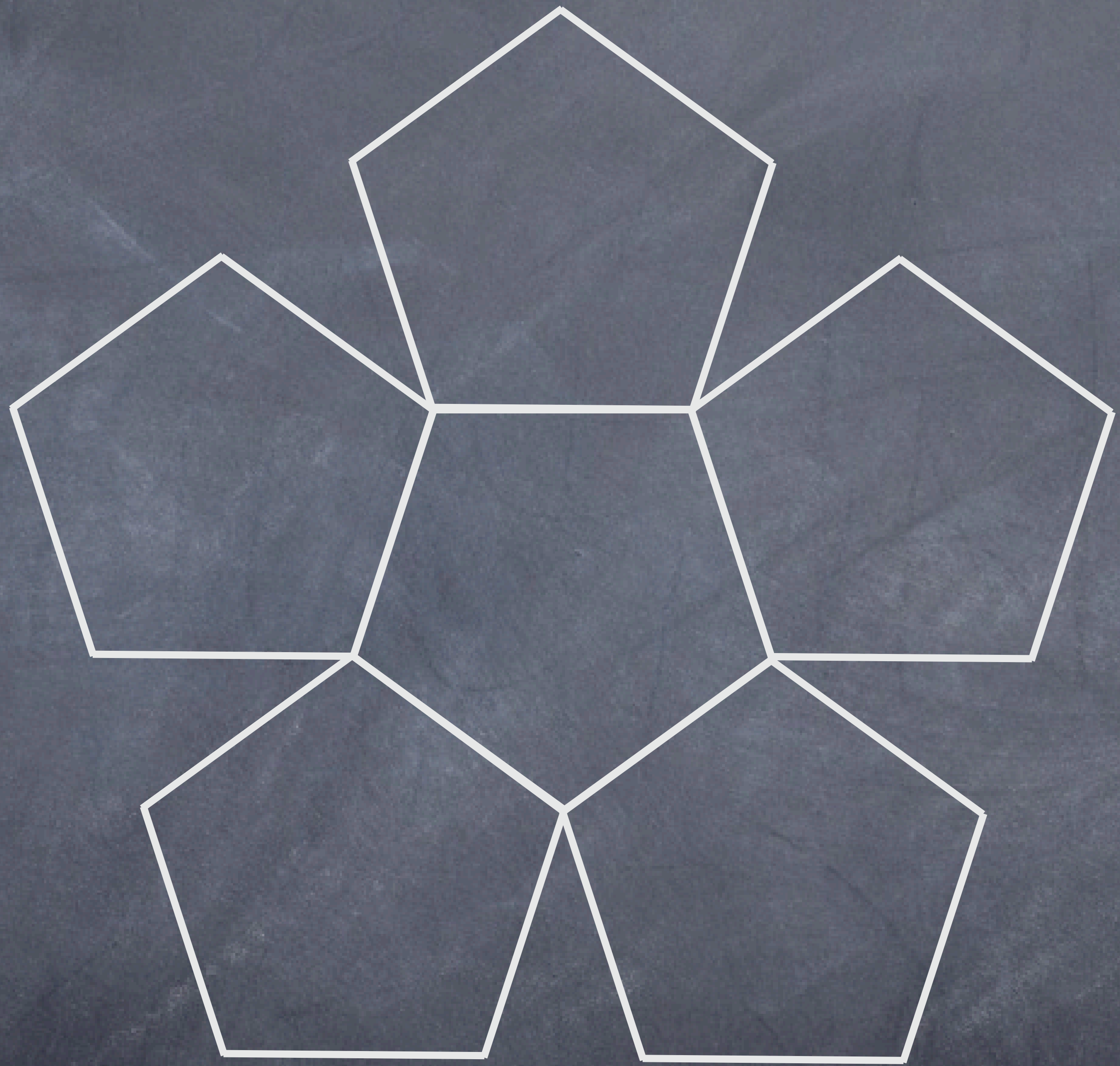
Why Triangle, Squares and Hexagons?



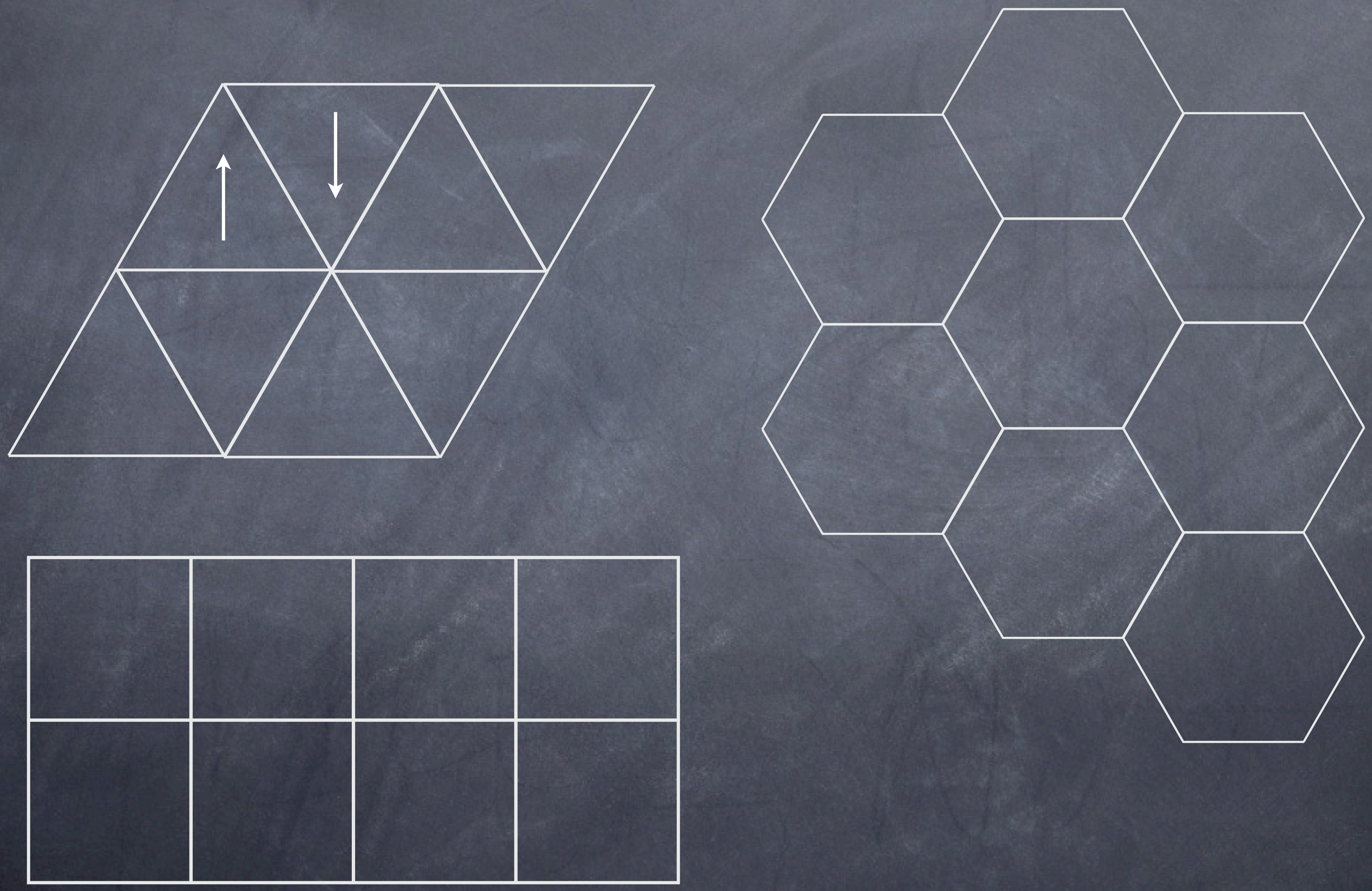


These are the three regular polygons that tile the plane.

What about pentagons?

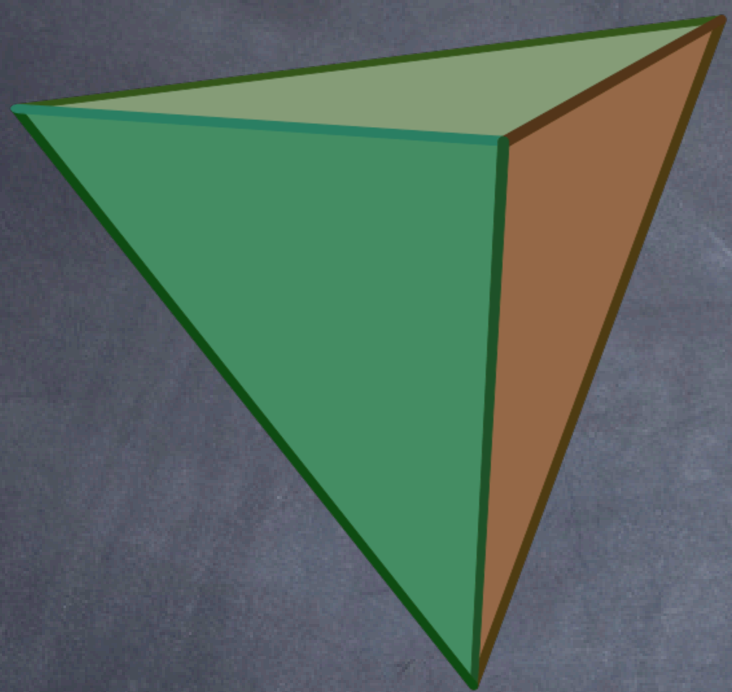


Note that the tiling of the plane can be accomplished with only translation of a single shape for squares and hexagons, but requires translation and rotation for triangles.

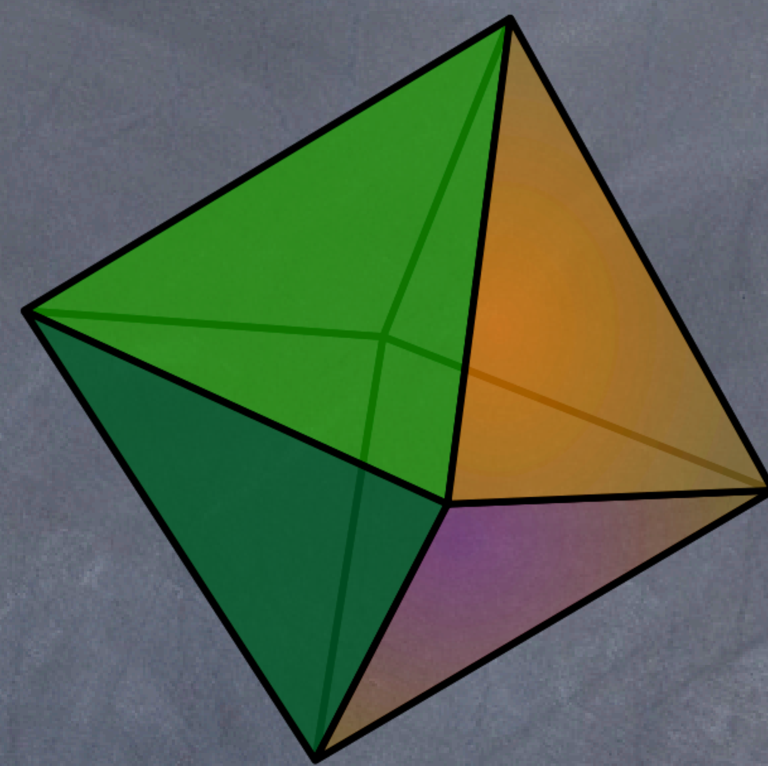


But we don't live on a plane surface ....

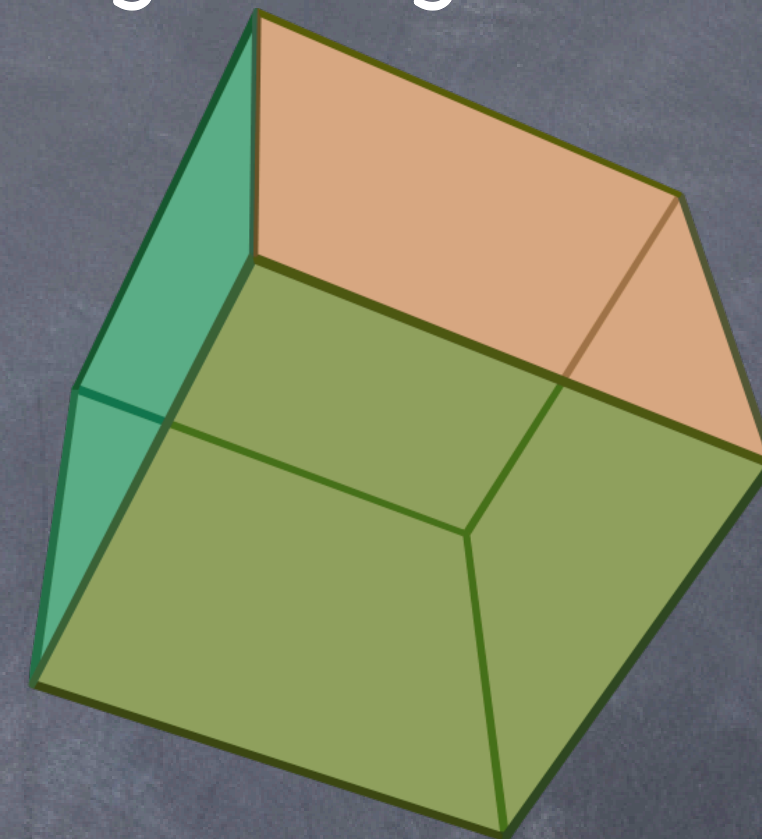
Getting from the plane to the sphere?
Projecting Platonic solids and natural gridding.



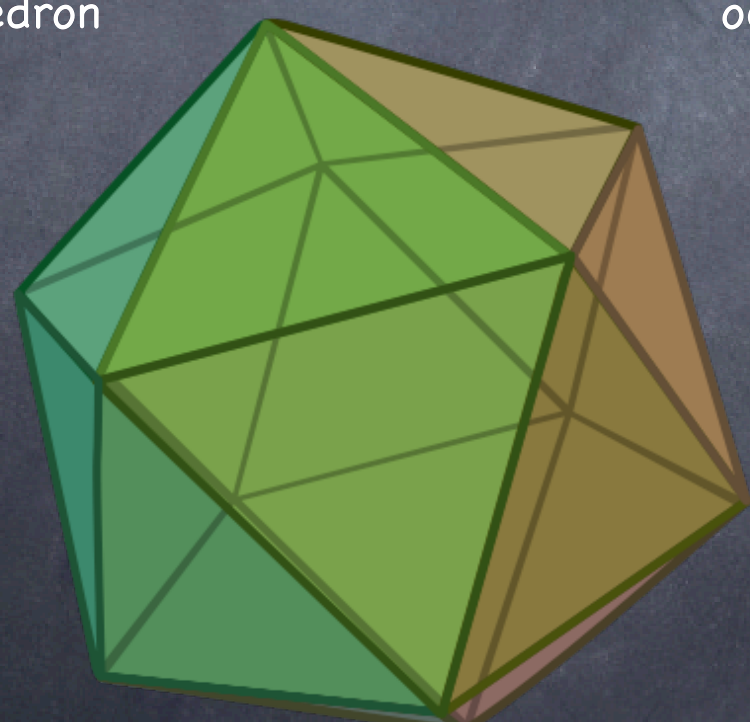
tetrahedron



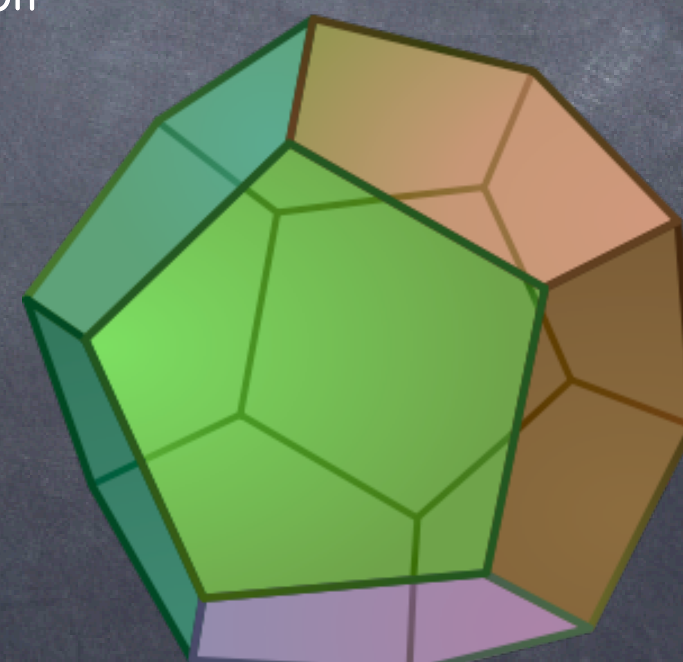
octahedron



hexahedron



icosahedron



dodecahedron

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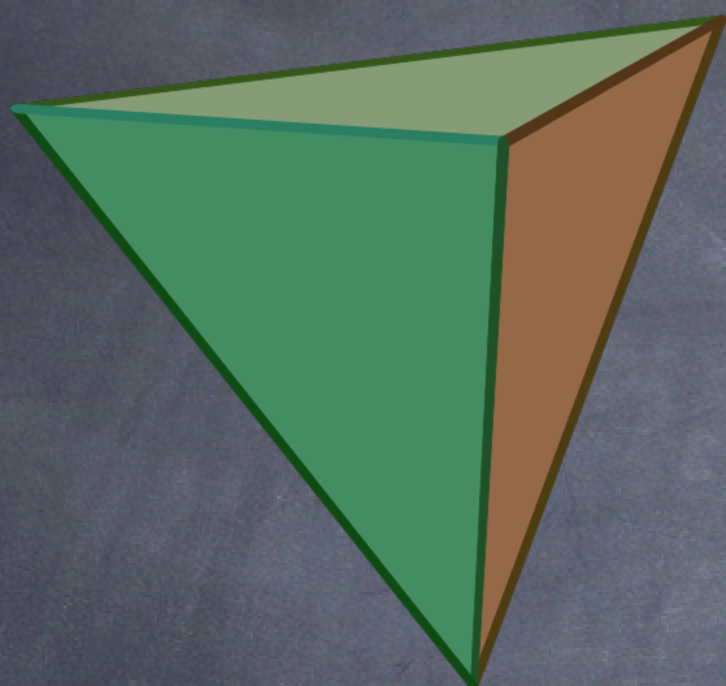
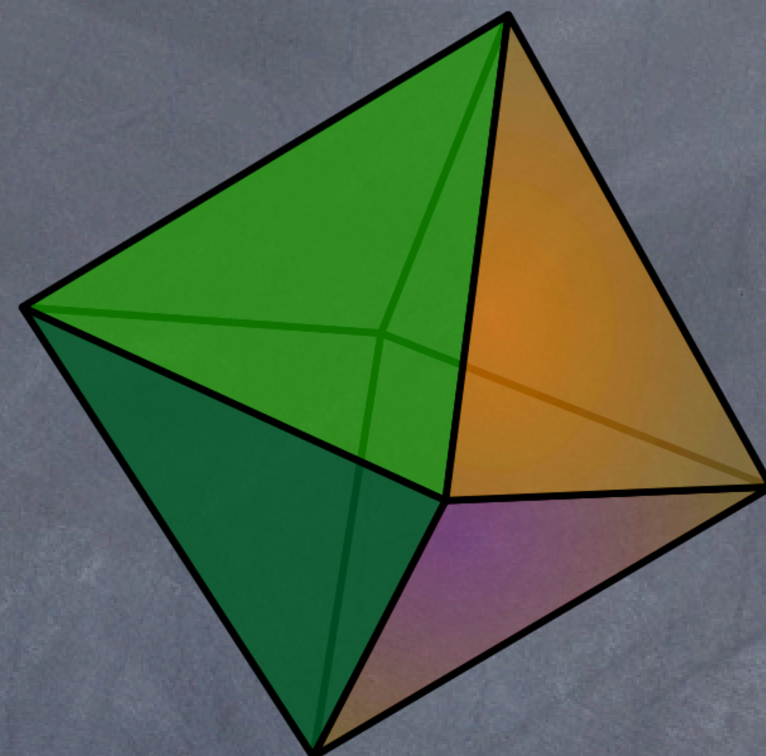
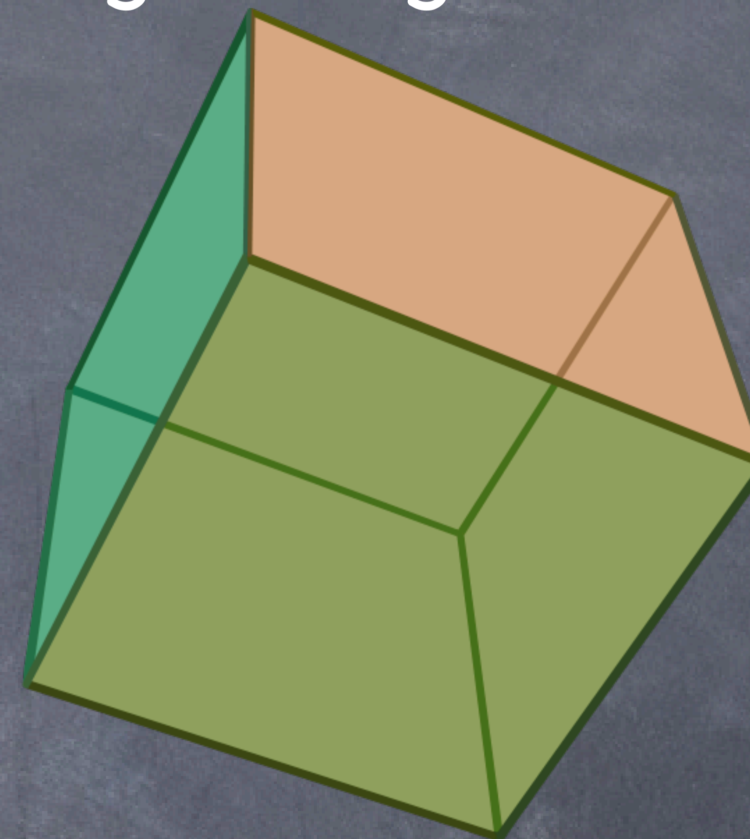


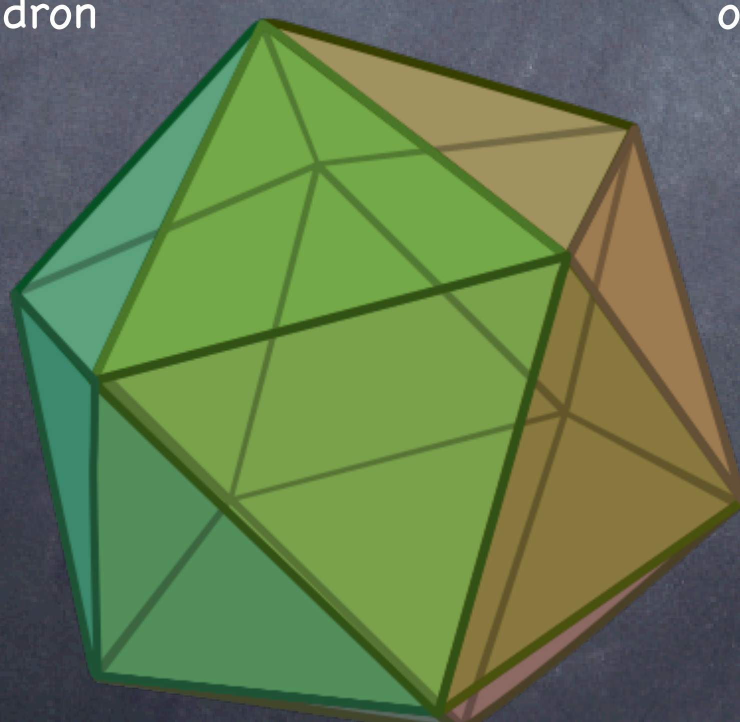
tetrahedron



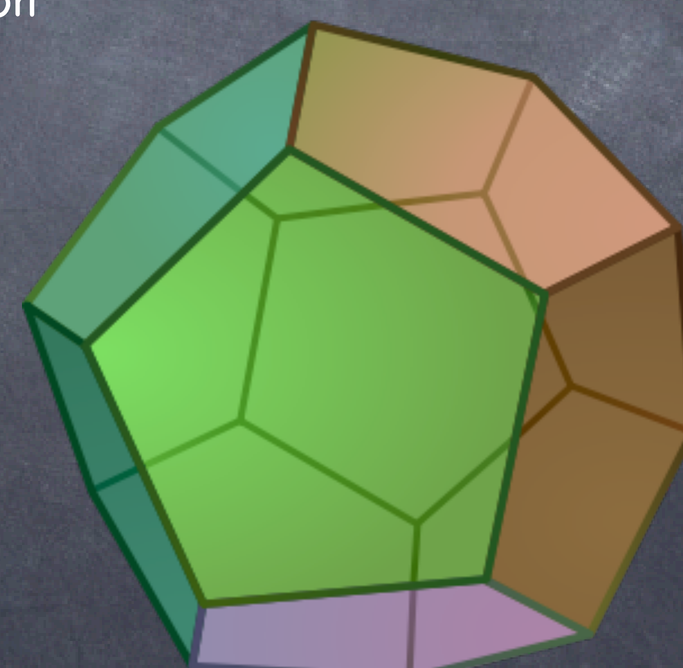
octahedron



hexahedron



icosahedron



dodecahedron

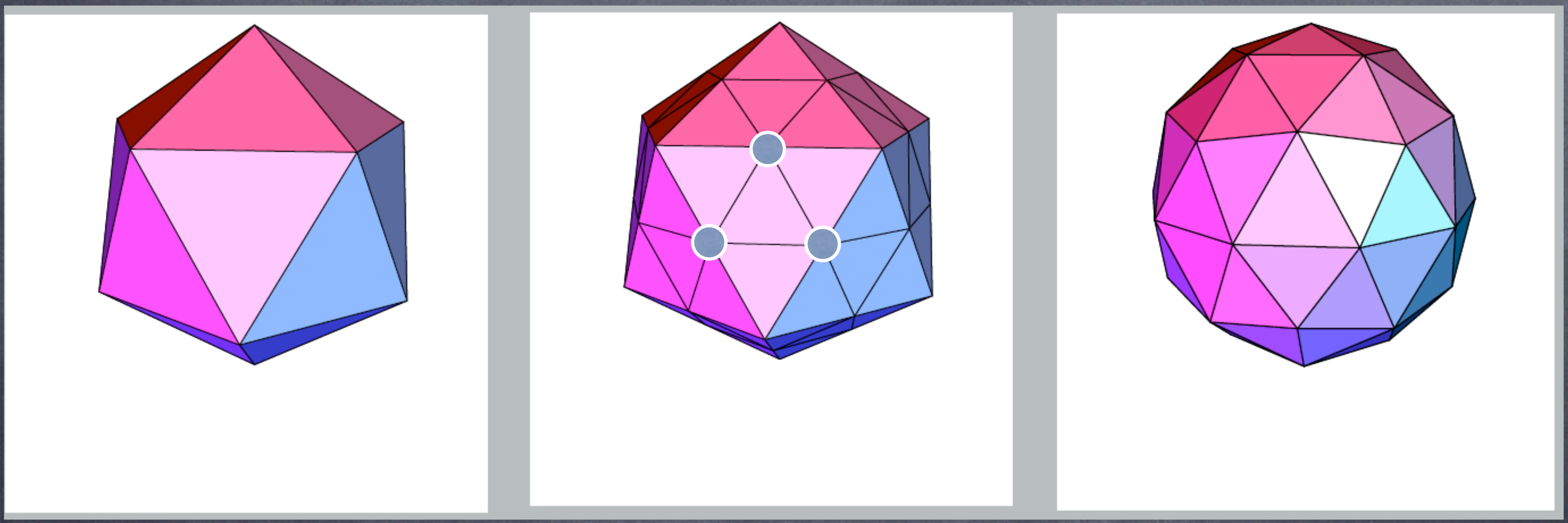
the only Platonic solid that does not admit a natural gridding system.

Example of natural gridding for icosahedron.

icosahedron

bisect each edge

project to unit sphere



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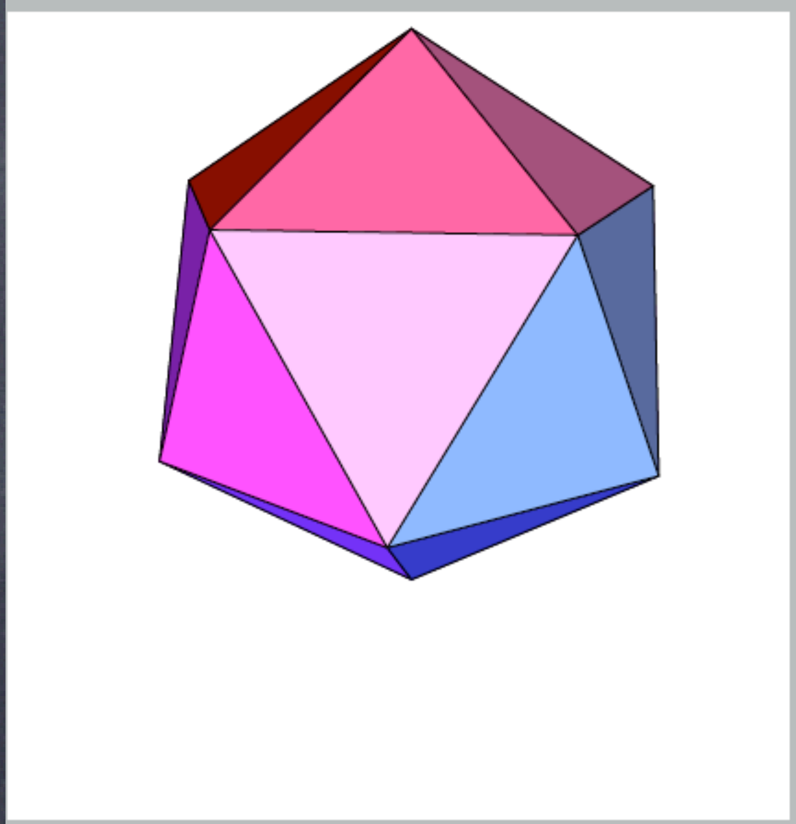
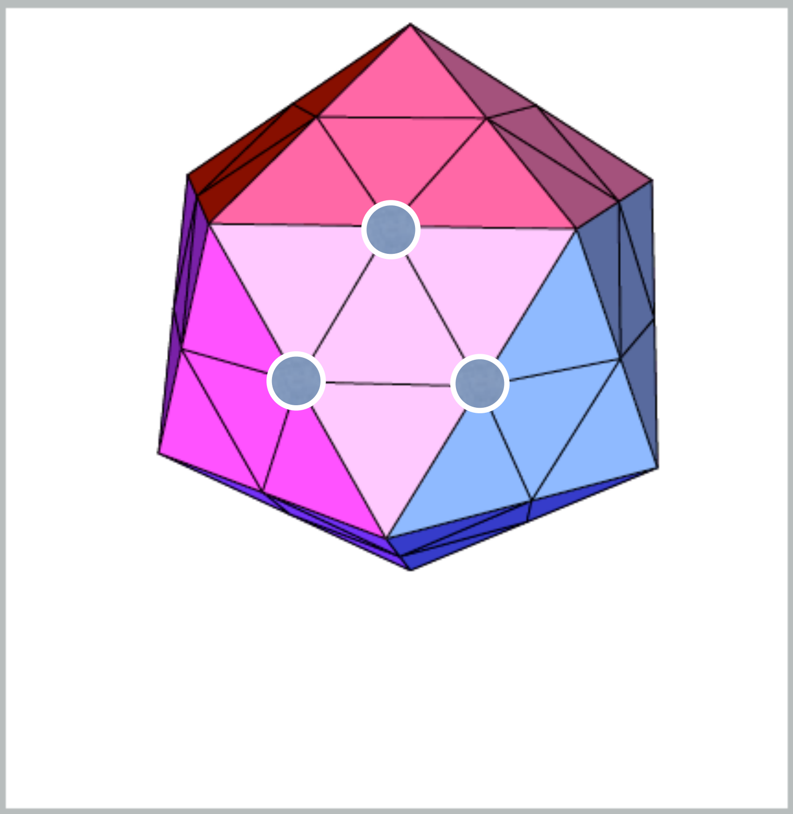
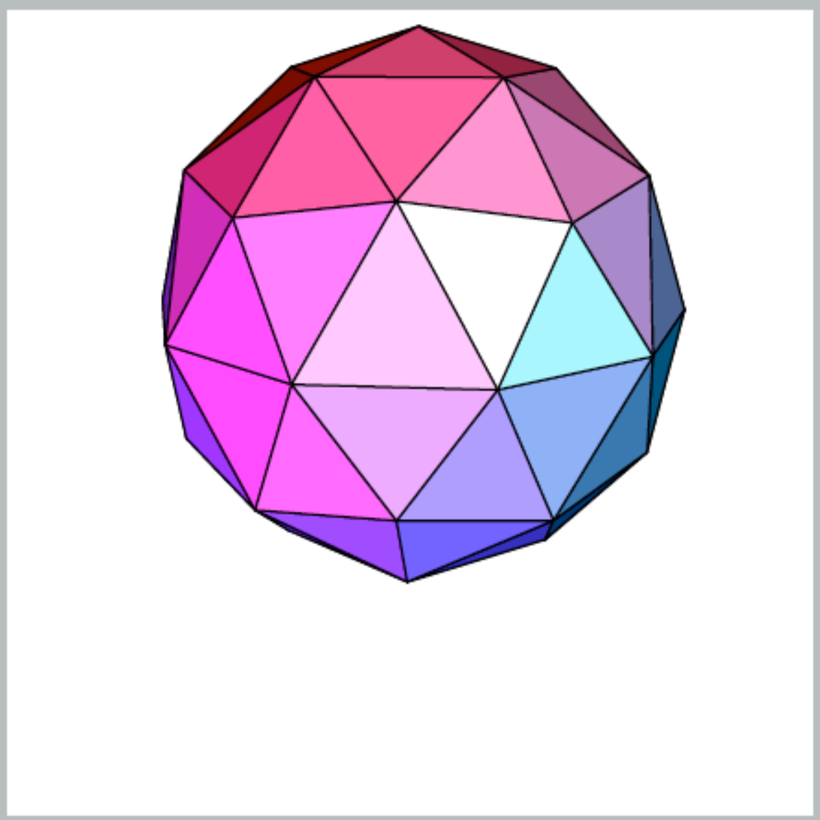


bisect each edge



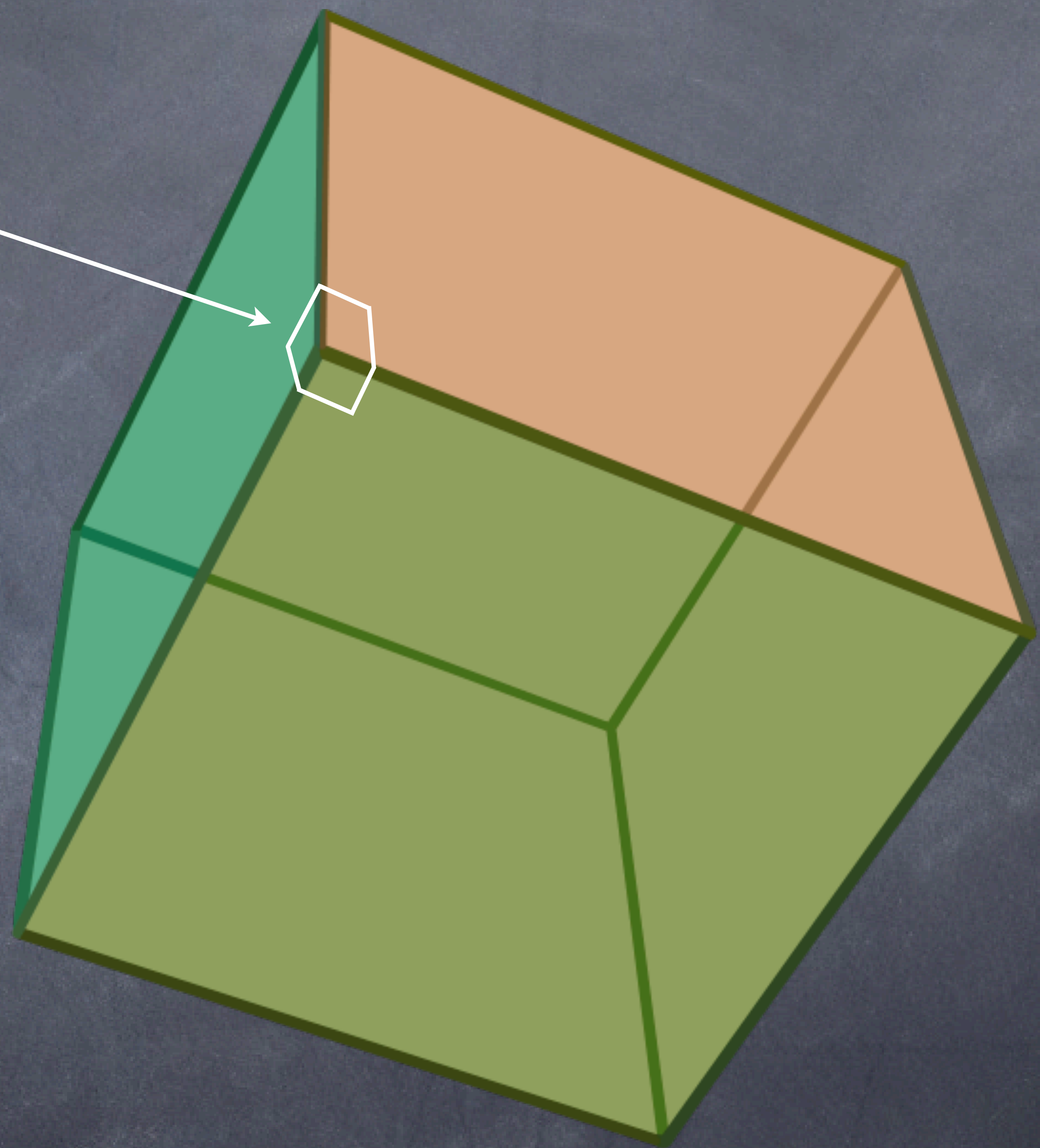
project to unit sphere



rinse and repeat .....

Angular deficiency and singularities.

Each corner is composed of three right angles (summing to 270 degrees). When projected to sphere, this corner will span 360 degrees. Each corner has an angular deficiency of 90 degrees.



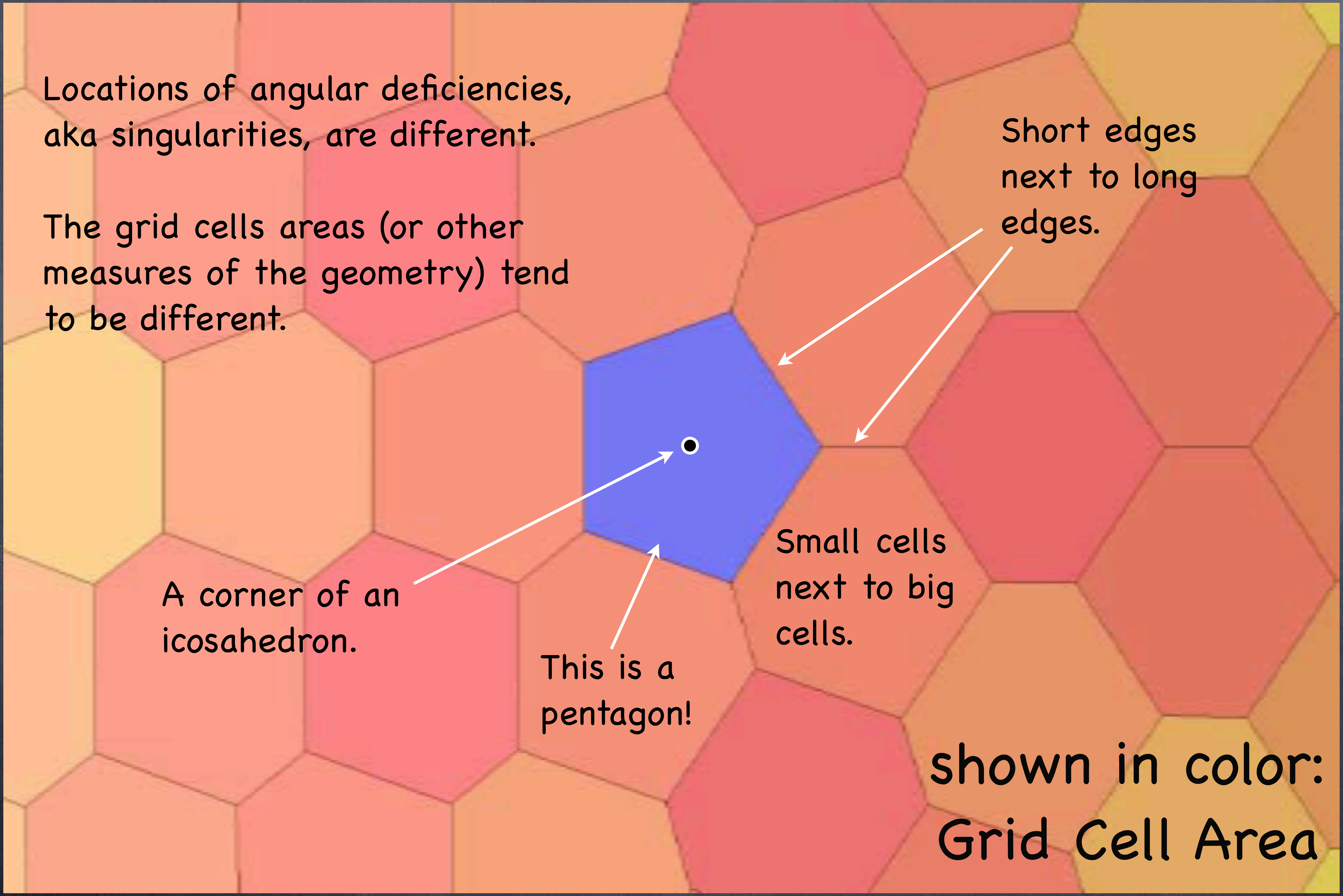
Regardless of the Platonic solid we choose, the total angular deficiency is 720 degrees. More corners imply a less severe singularity at each corner.

From a practical perspective, are these singularities a problem?

Locations of angular deficiencies, aka singularities, are different.

Indexing is different. Note that at the singularity the corner has only three connected edges (just like the solid cube).

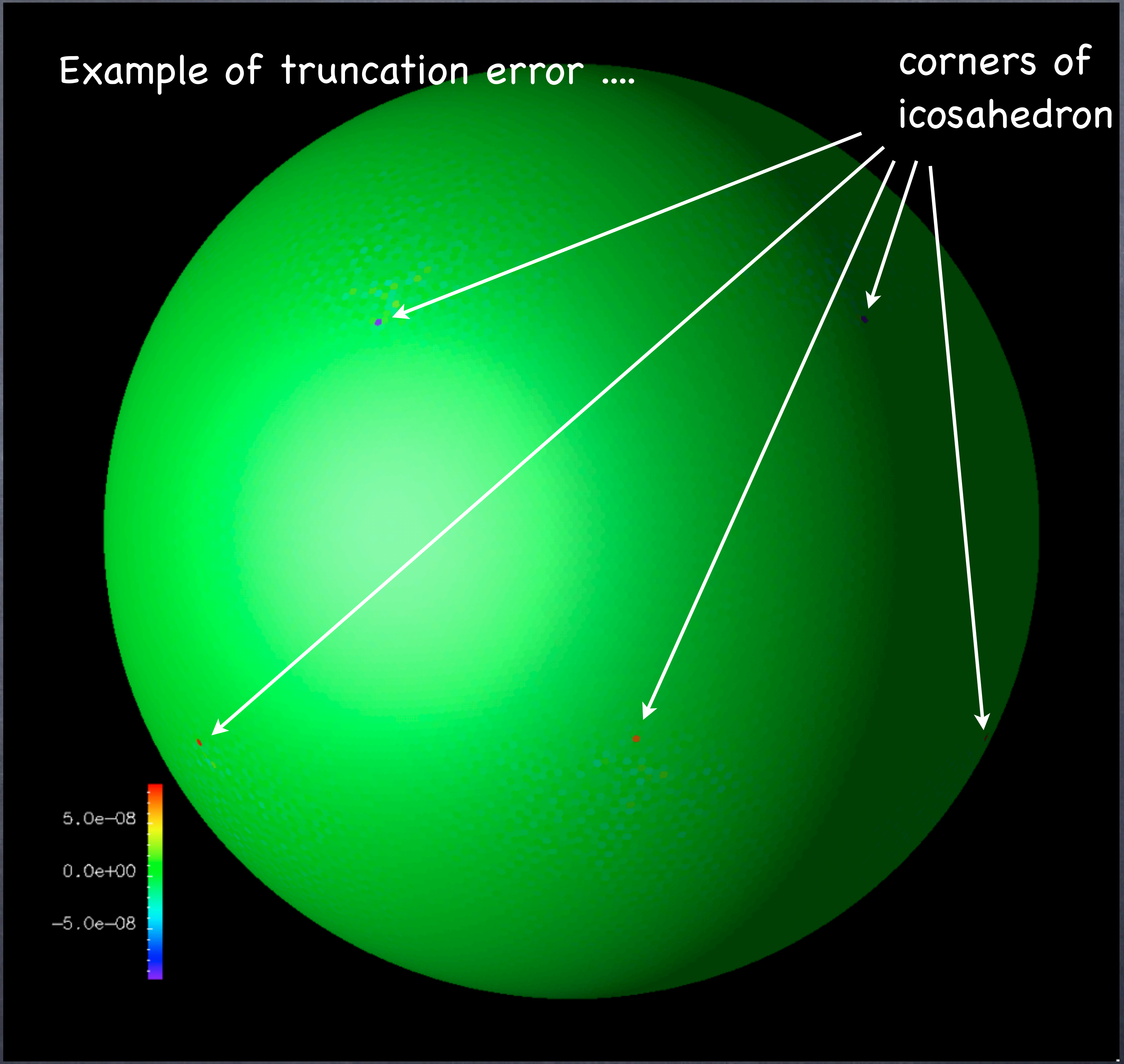
From a practical perspective, are these singularities a problem?



From a practical perspective, are these singularities a problem?

Singularities are regions of grid distortion, as a result truncation error is generally largest at these locations.

This is a much more severe problem when using low-order operators (FV) then with high-order operators (spectral elements).



So mapping to the sphere is not without its issues. The issue of angular deficiency and the consequences that follow will always have to be dealt with.

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When assessing the Platonic solids solely on the criteria of singularity strength and access to a natural gridding system, the ranking would be (from best to worst): icosahedron, hexahedon, octahedron, tetrahedron.

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Of these four, the icosahedon and hexahedron have seen significant interest.

A closer comparison of triangles, quads and hexagons is still a useful exercise.

Let's start out with a qualitative assessment based on isotropy

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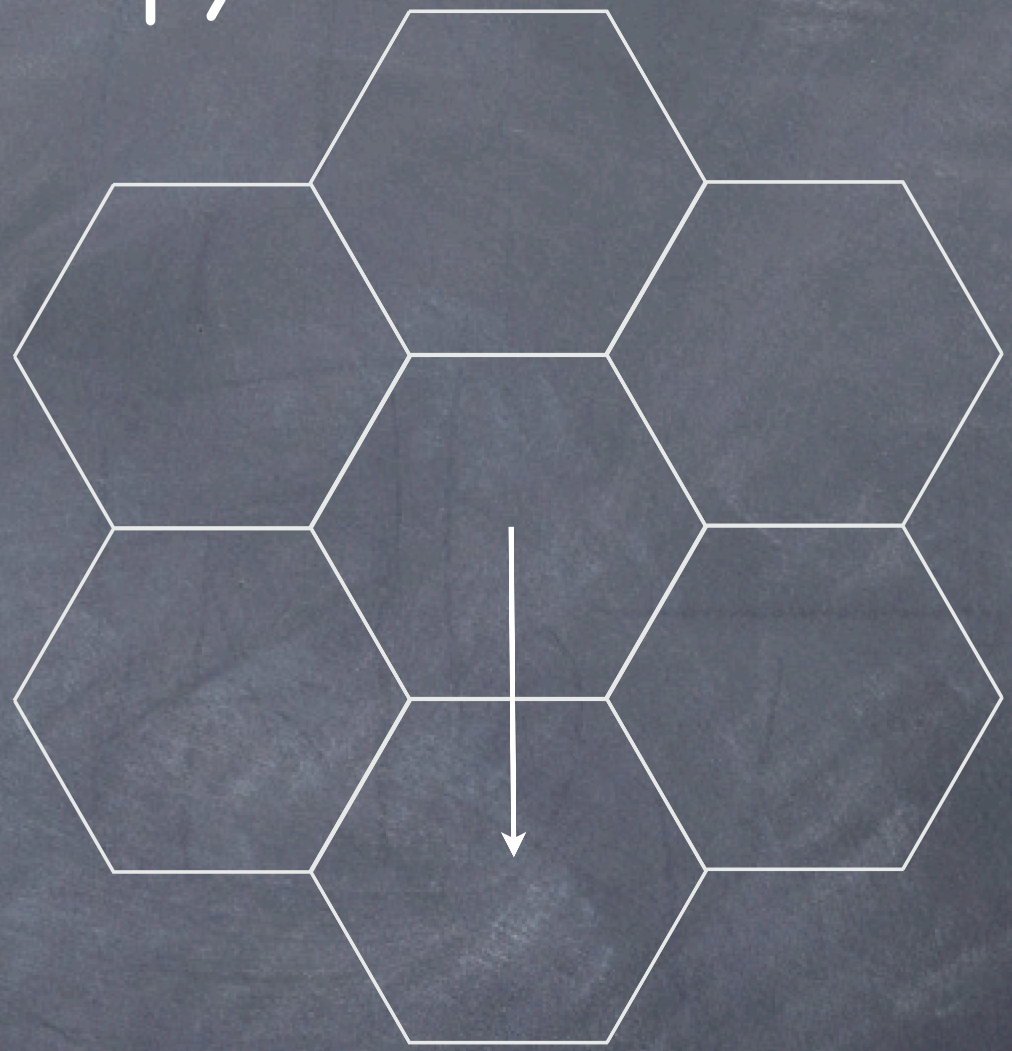

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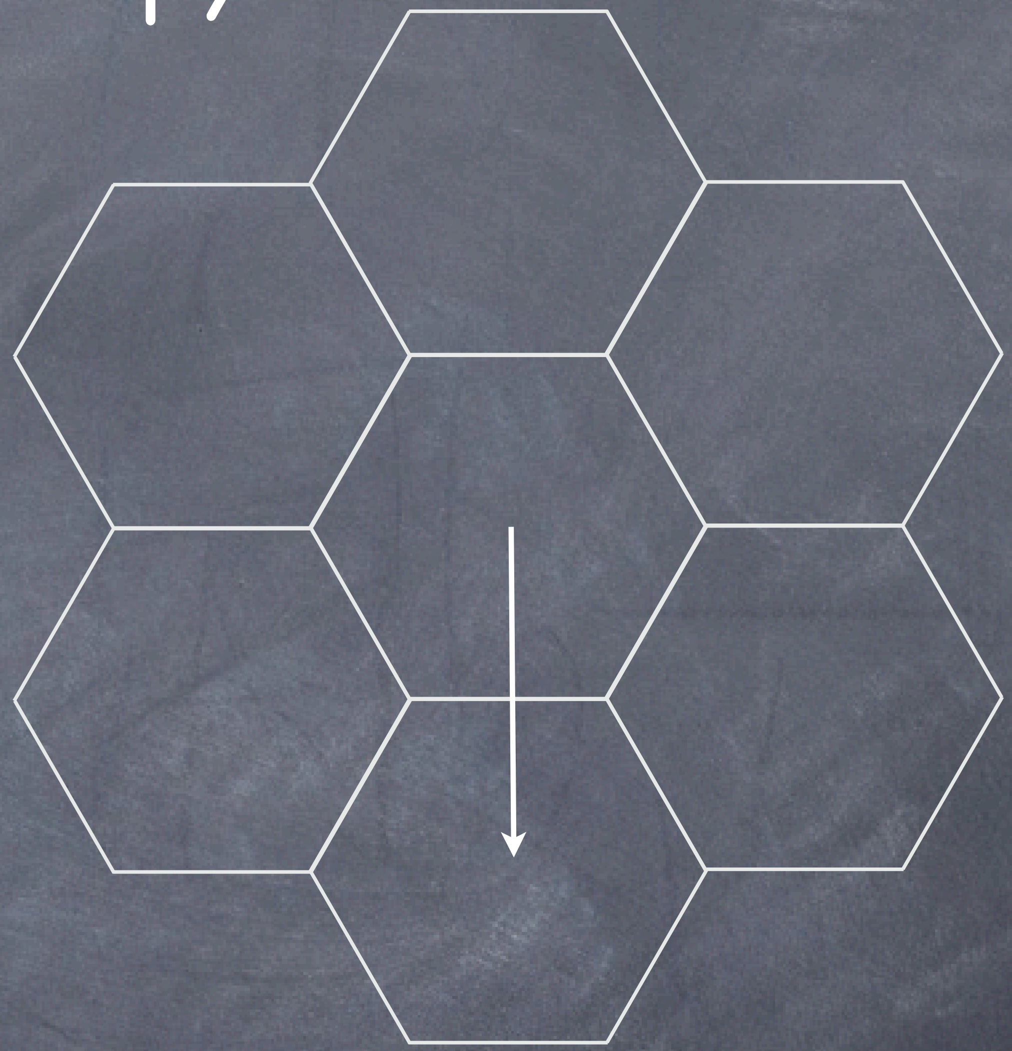

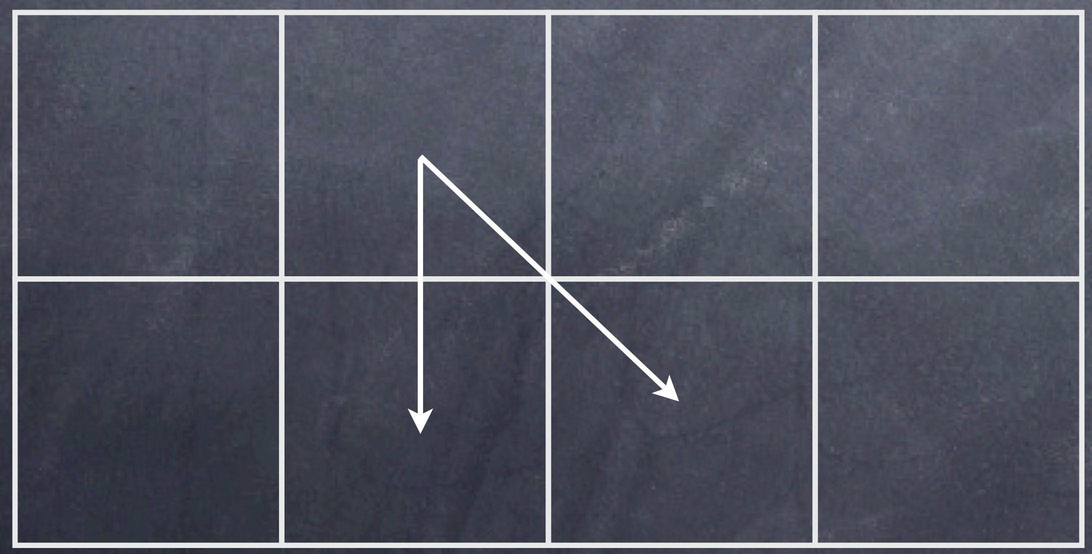

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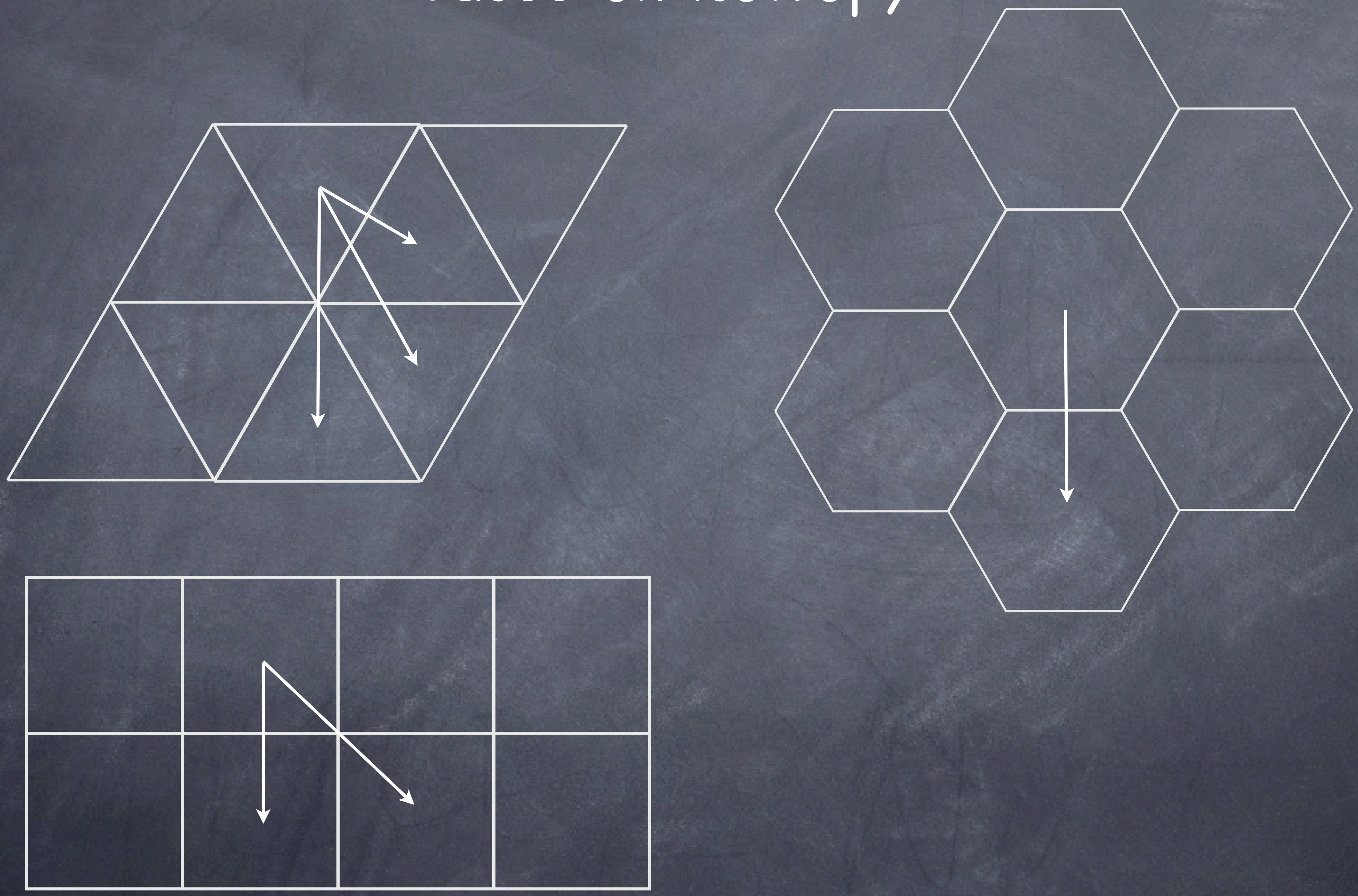

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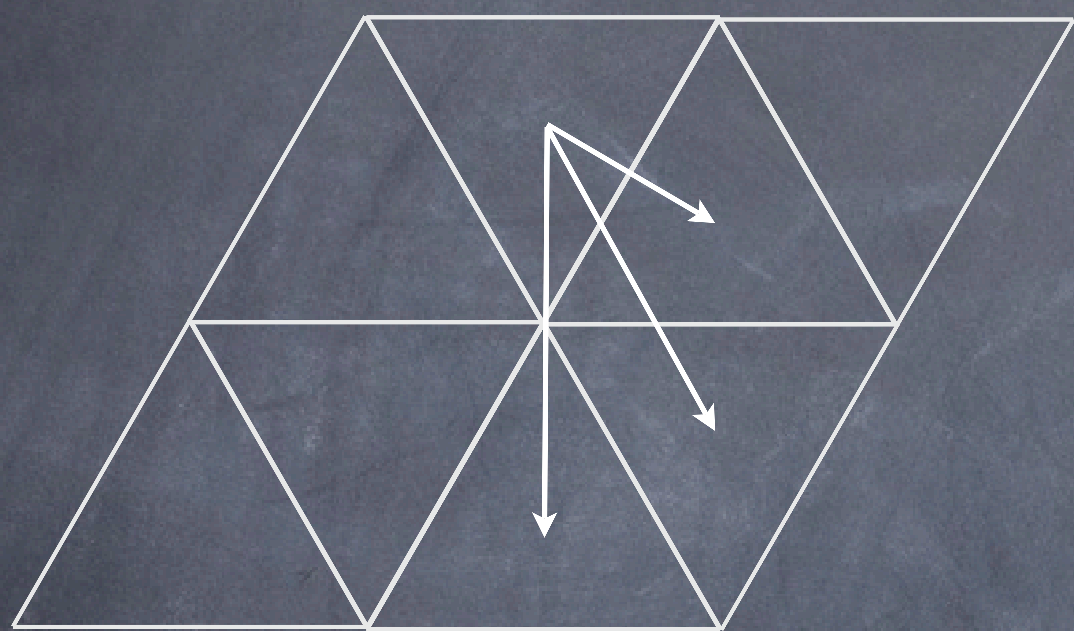
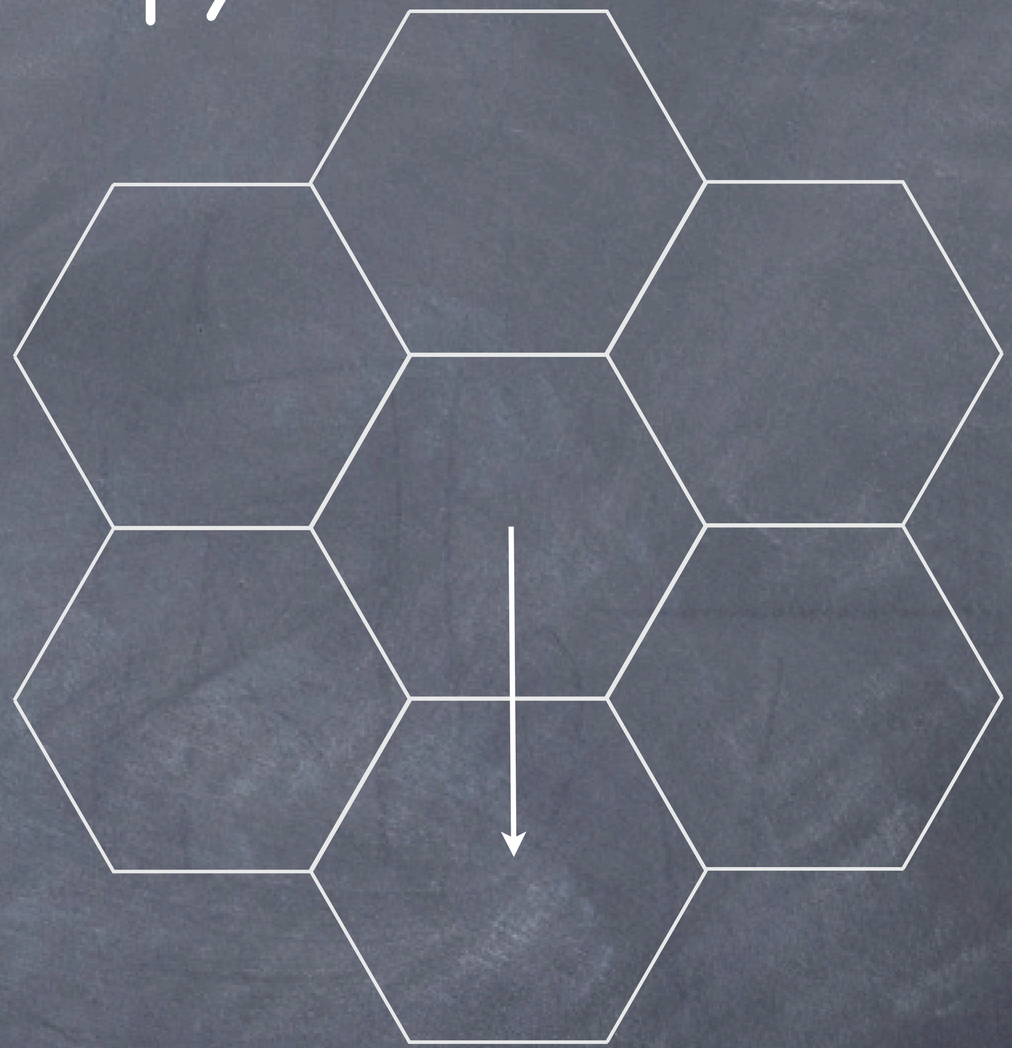
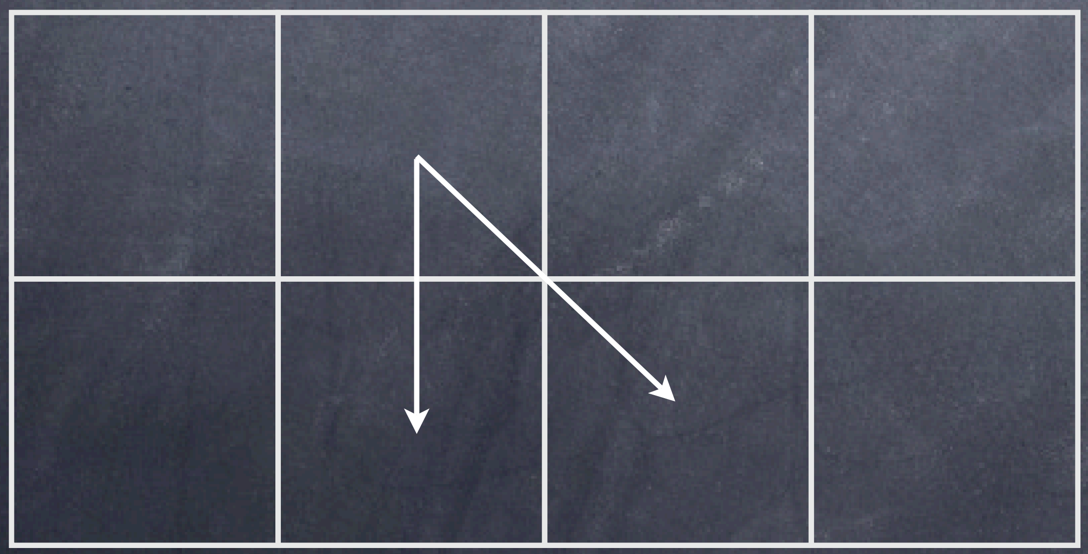


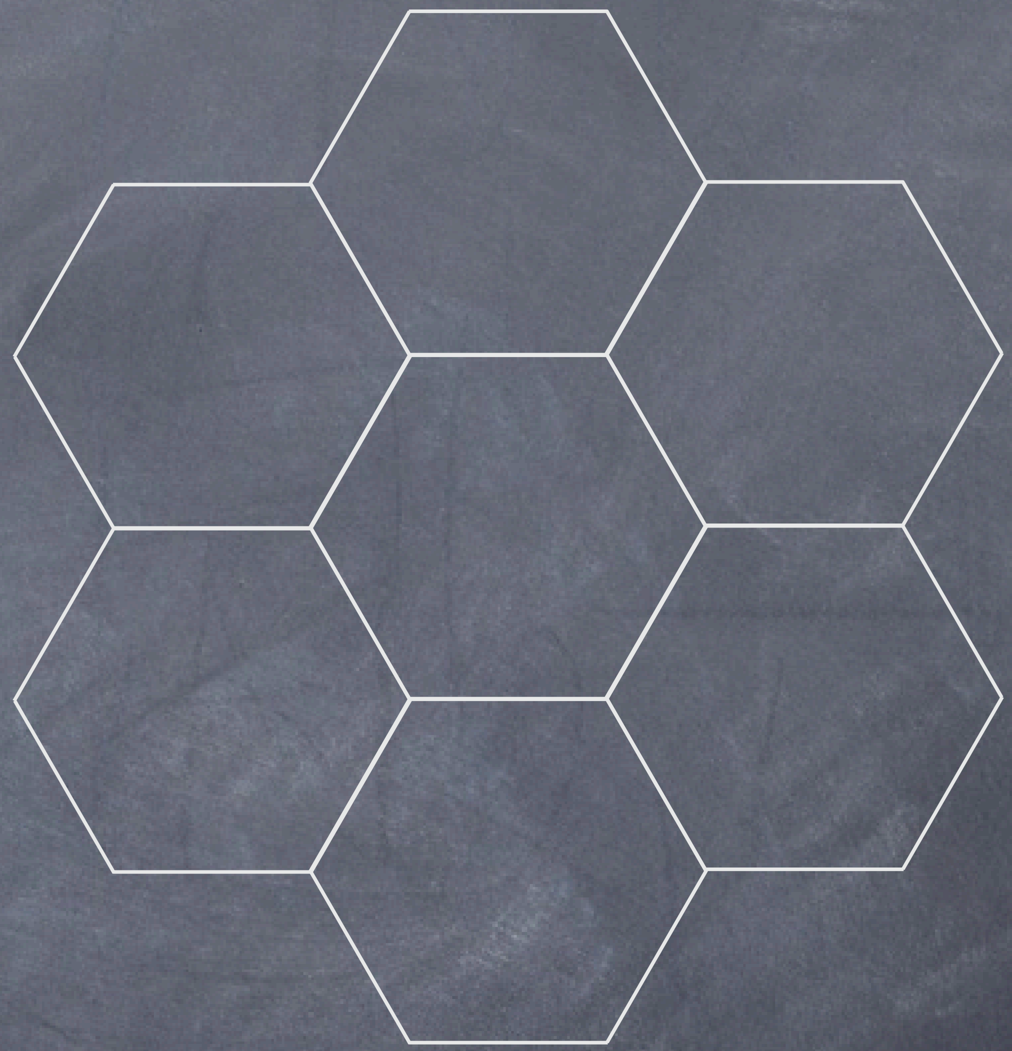




Neighbors:

- Hexagons: one kind
- Quads: two kinds
- Triangles: three kinds

Impacts of isotropy ....



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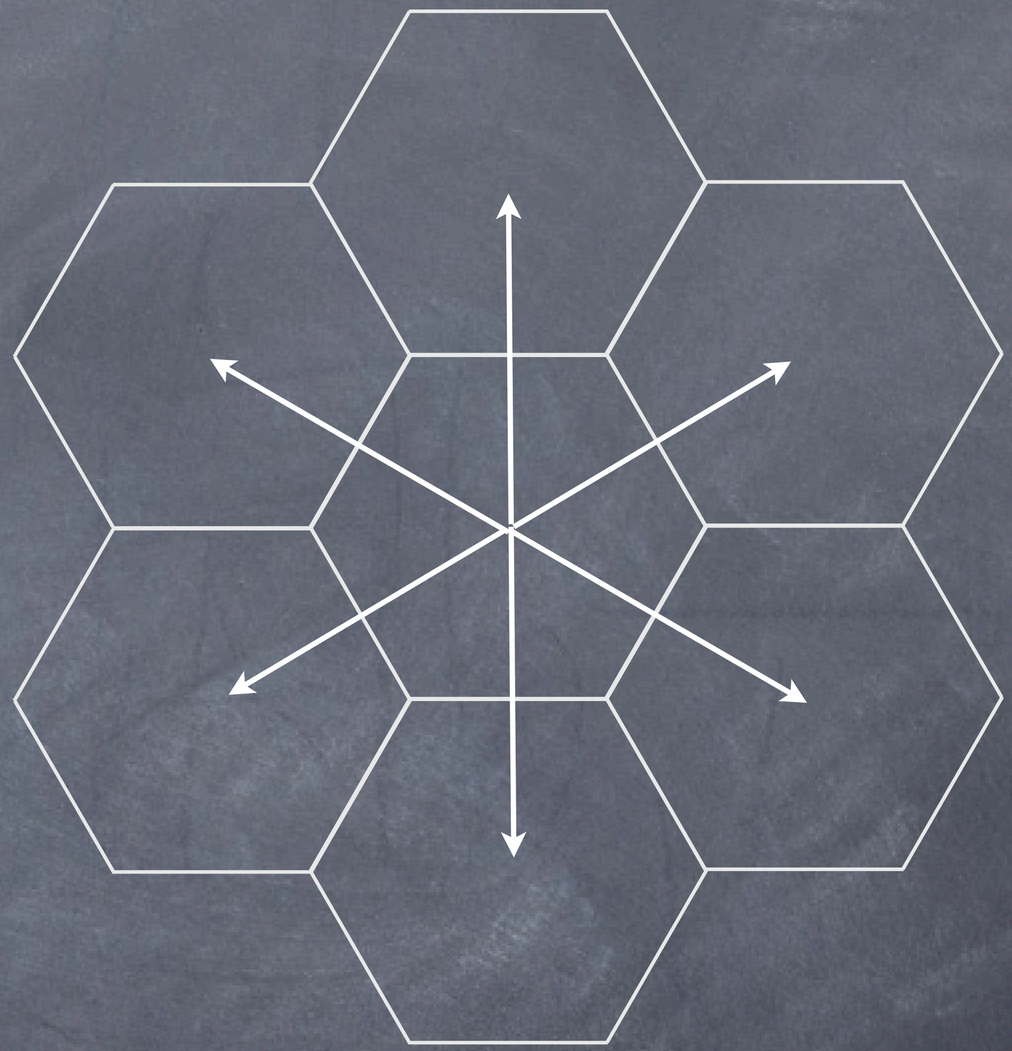

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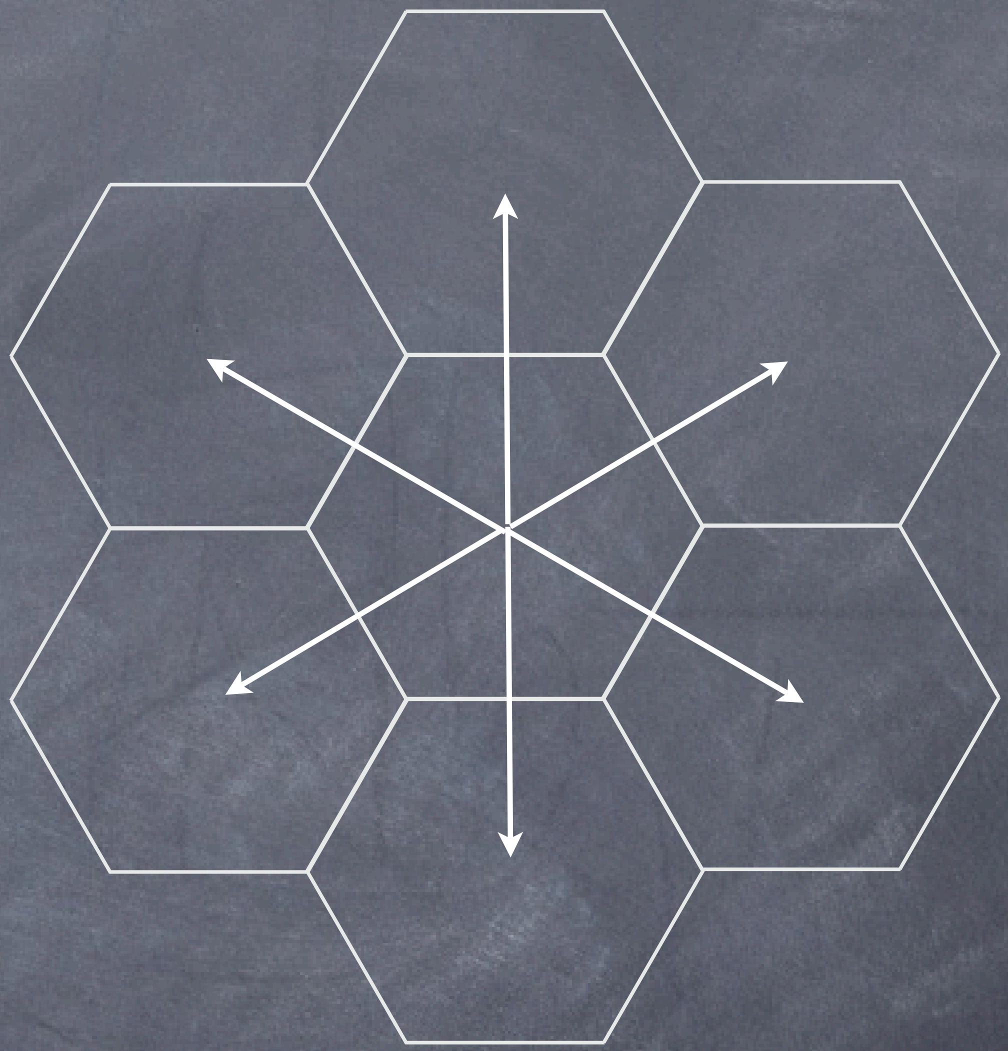

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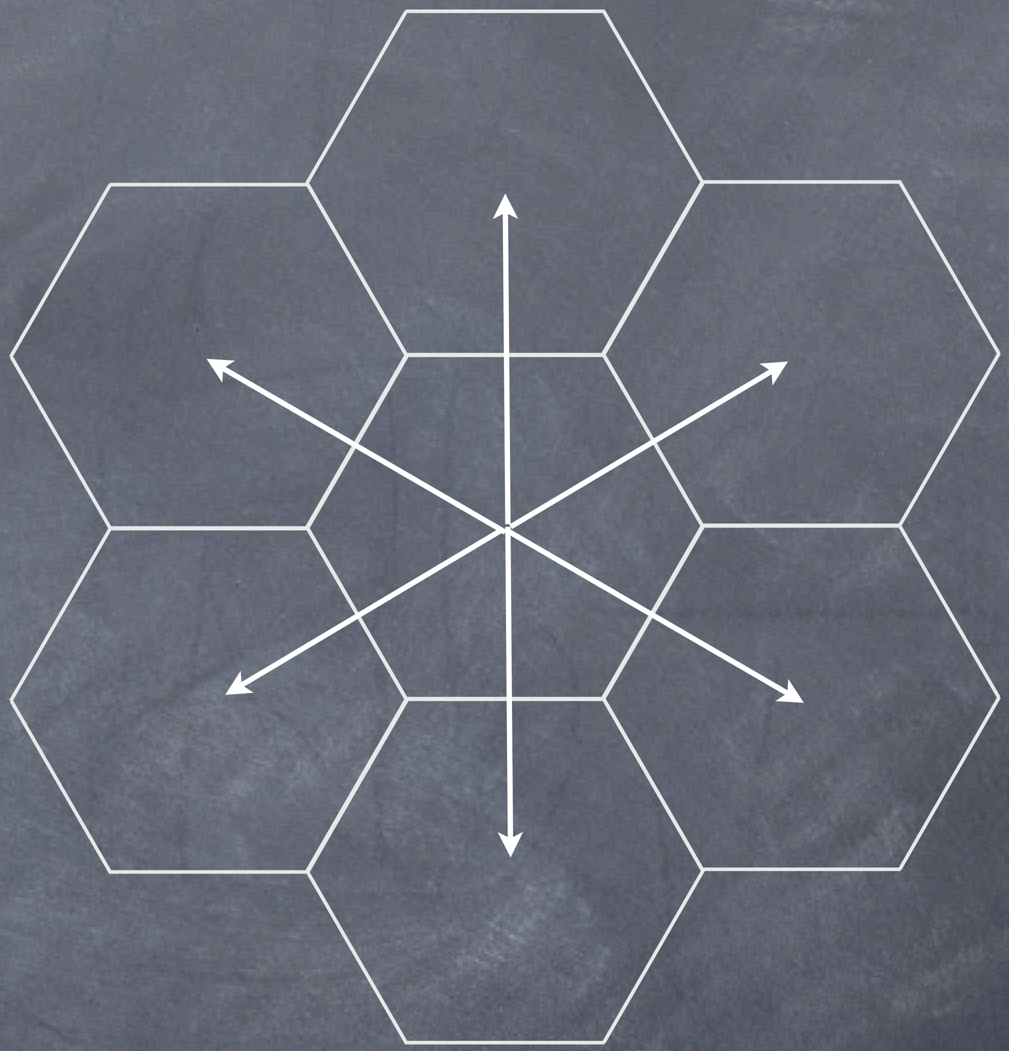

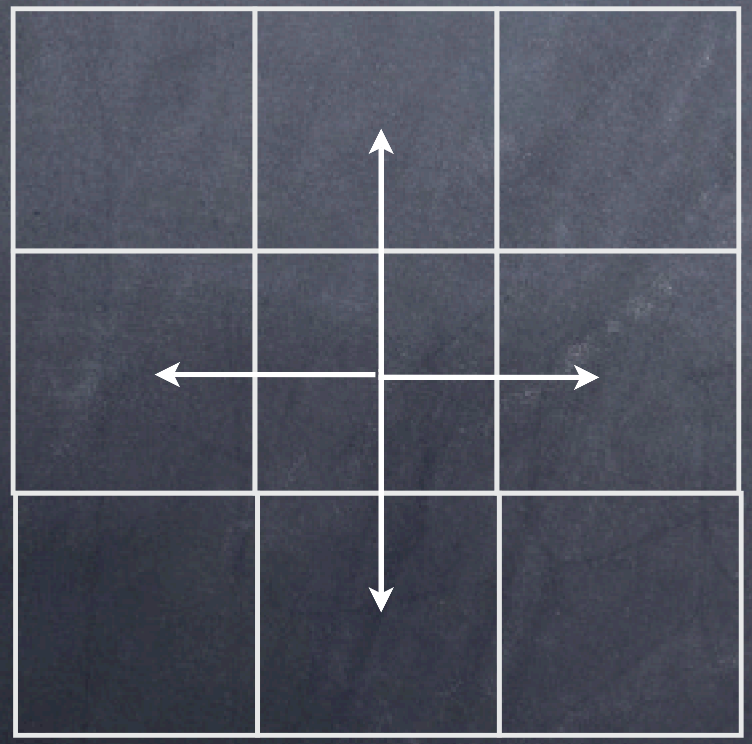

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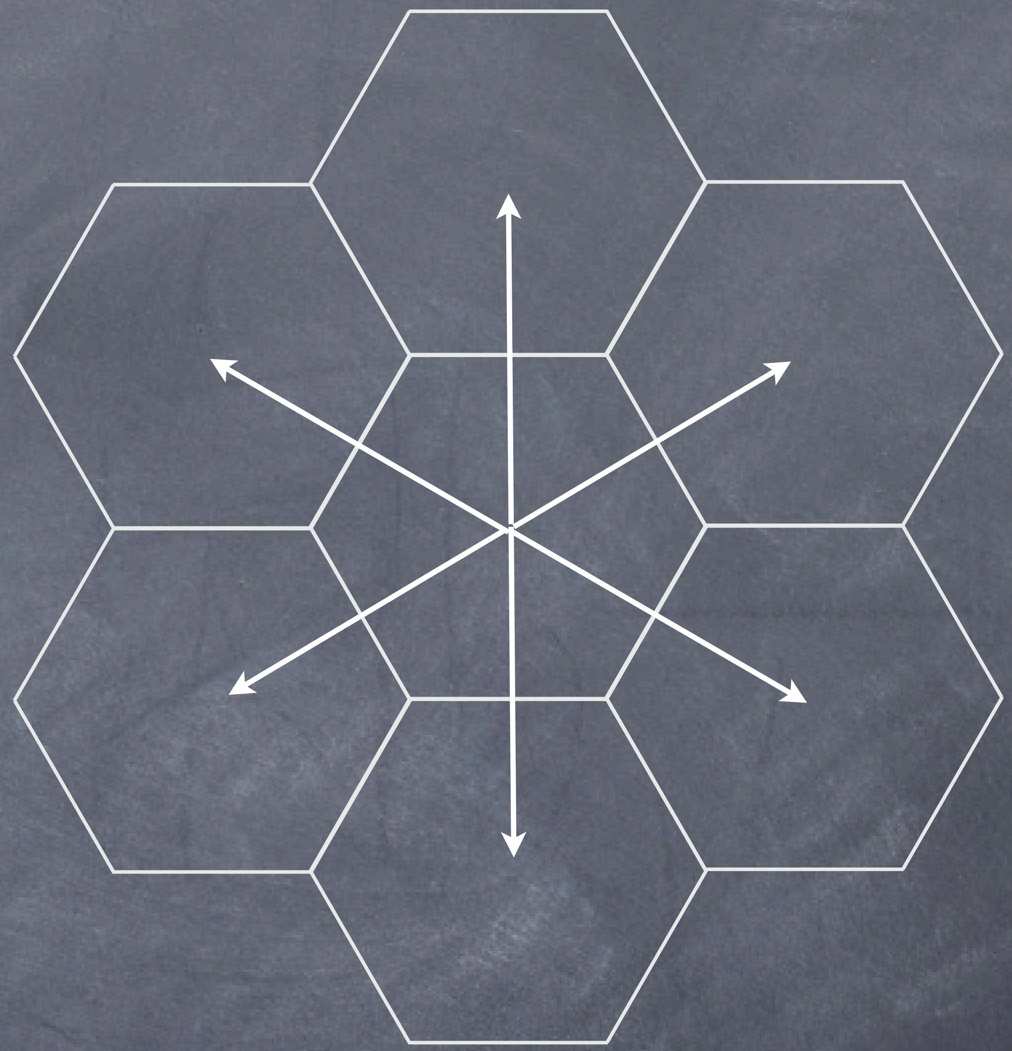

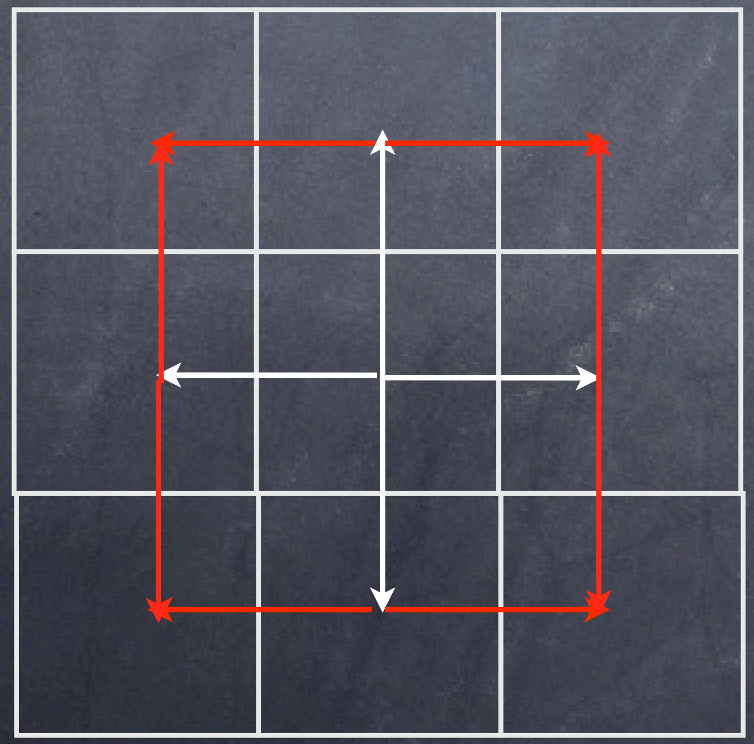

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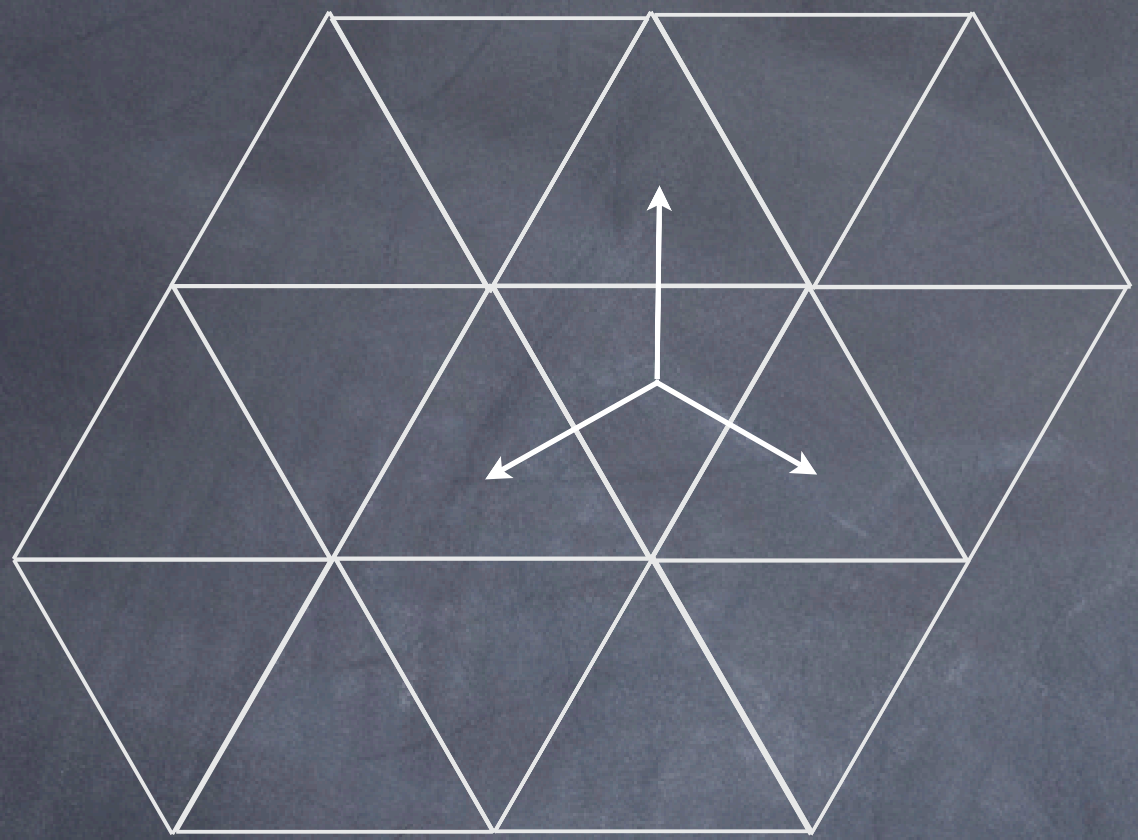

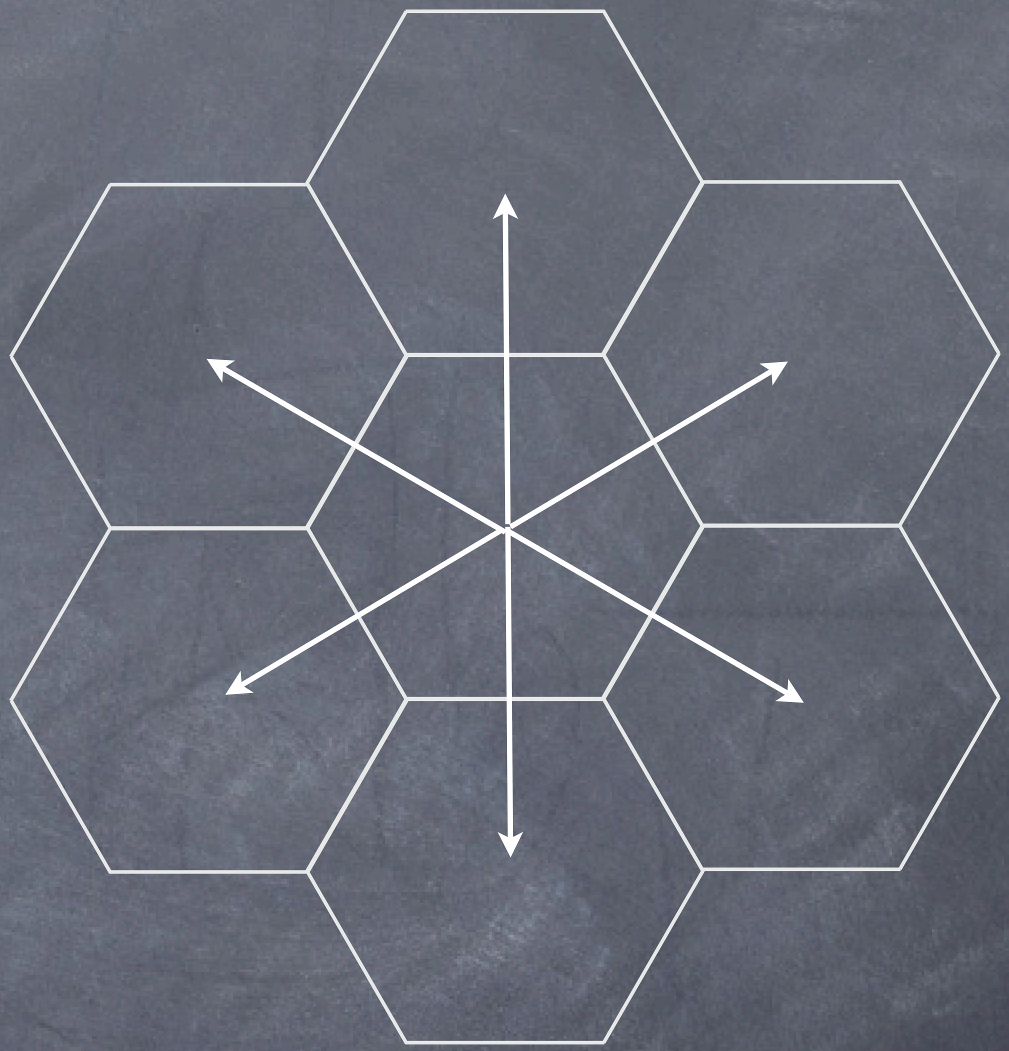

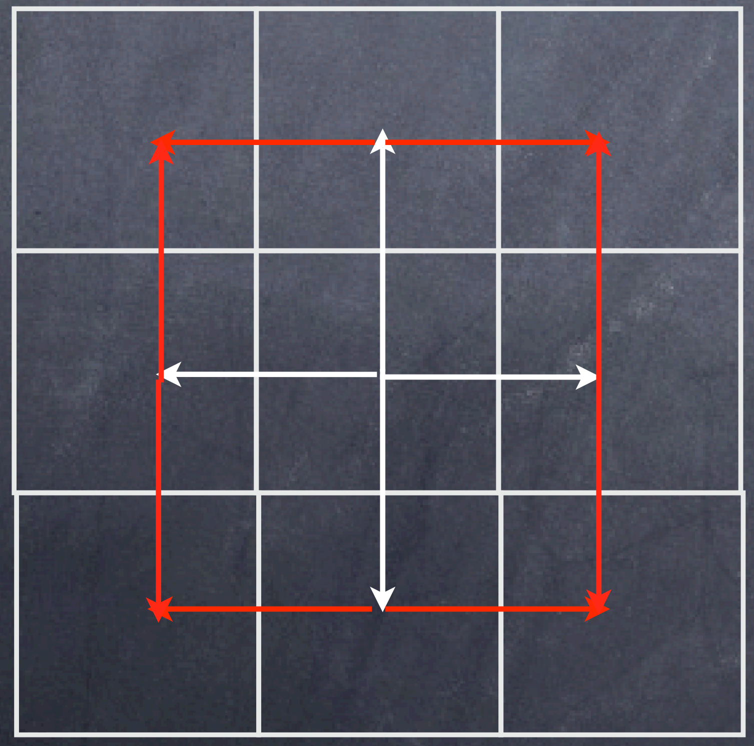

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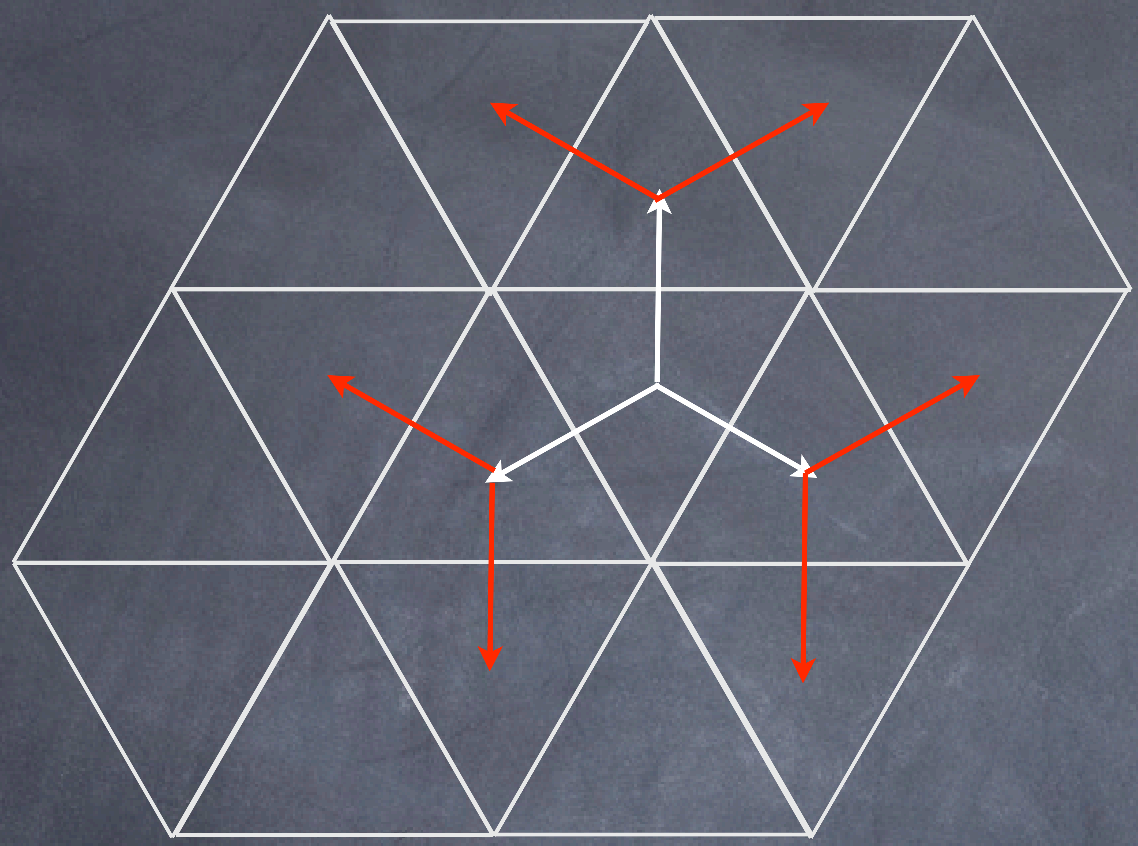

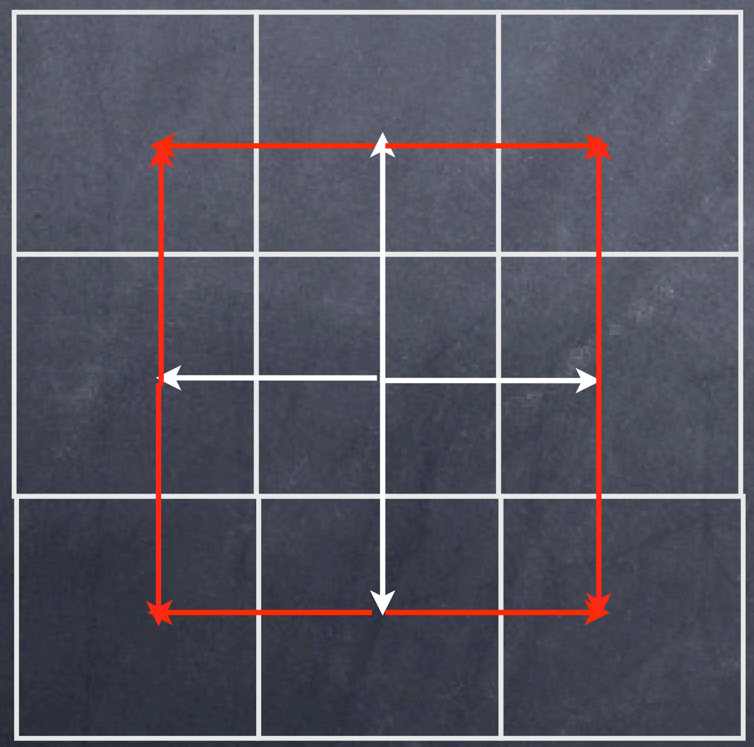

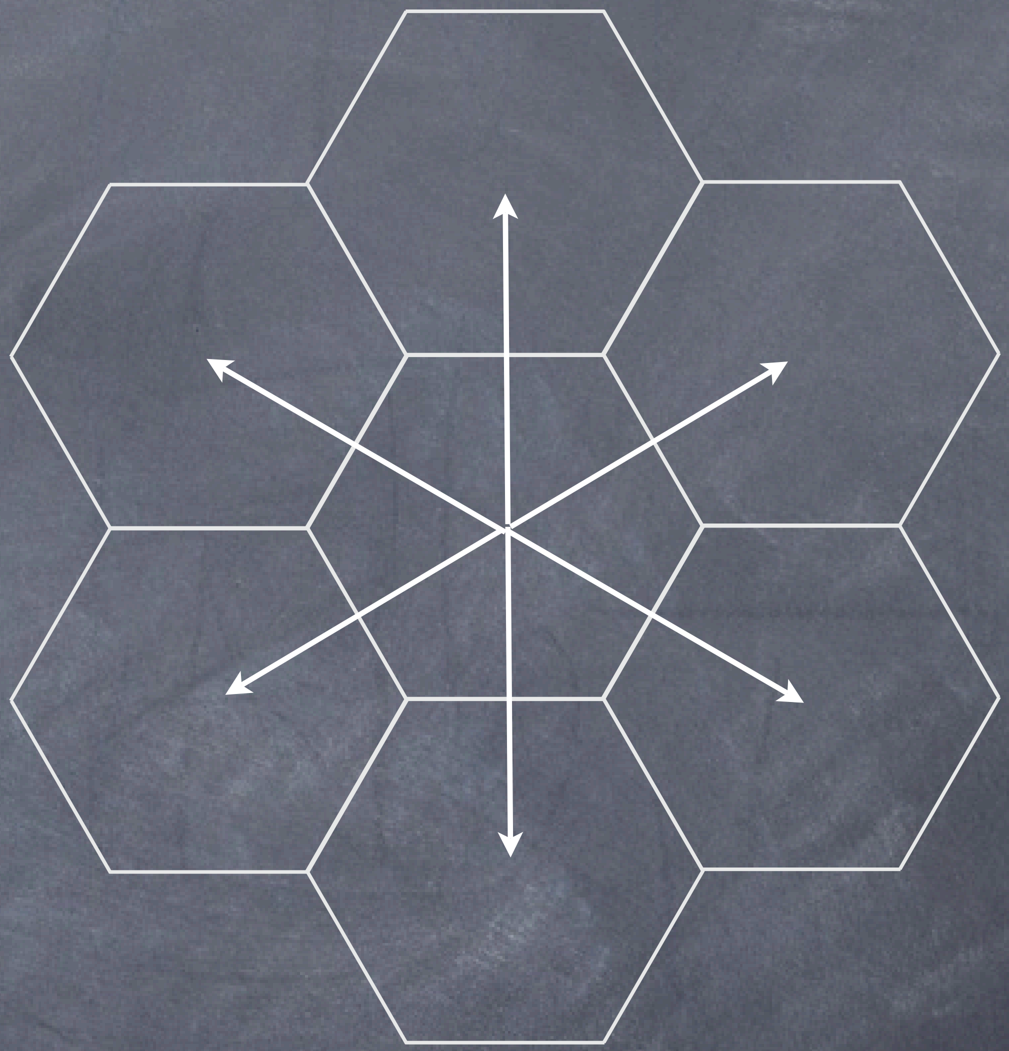

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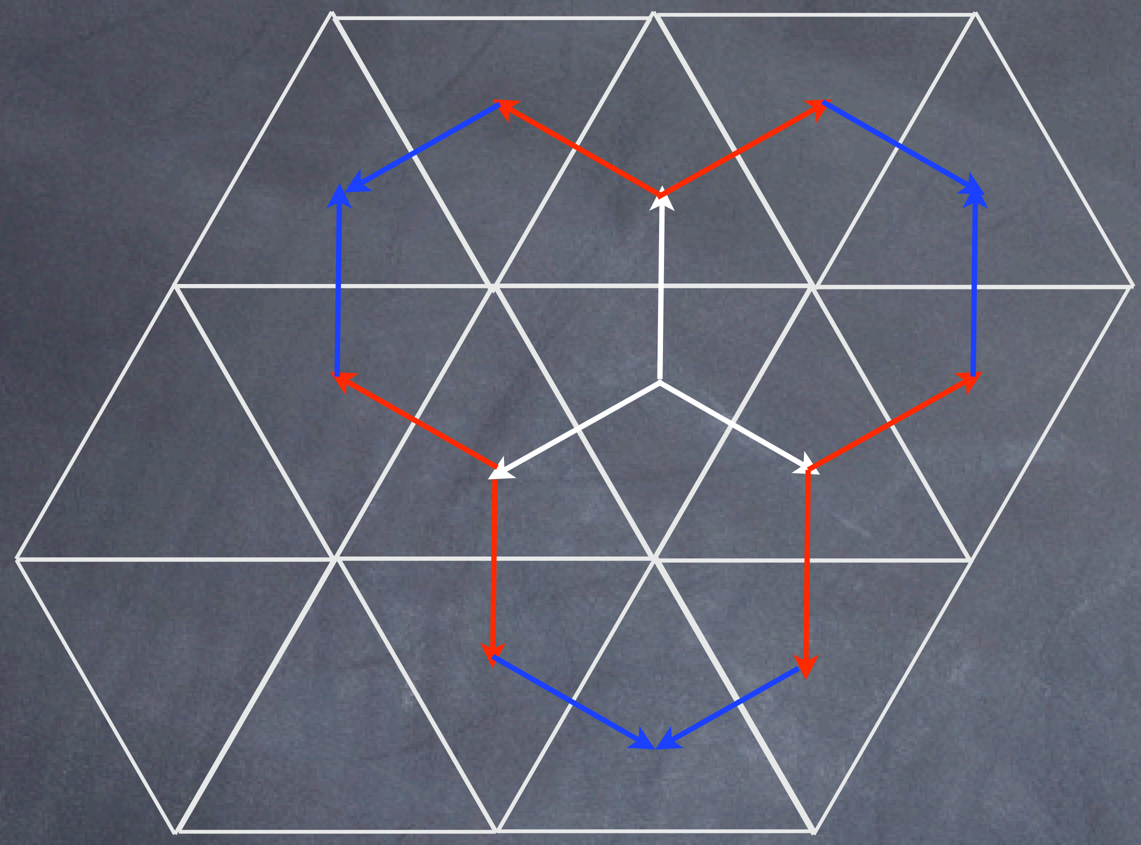

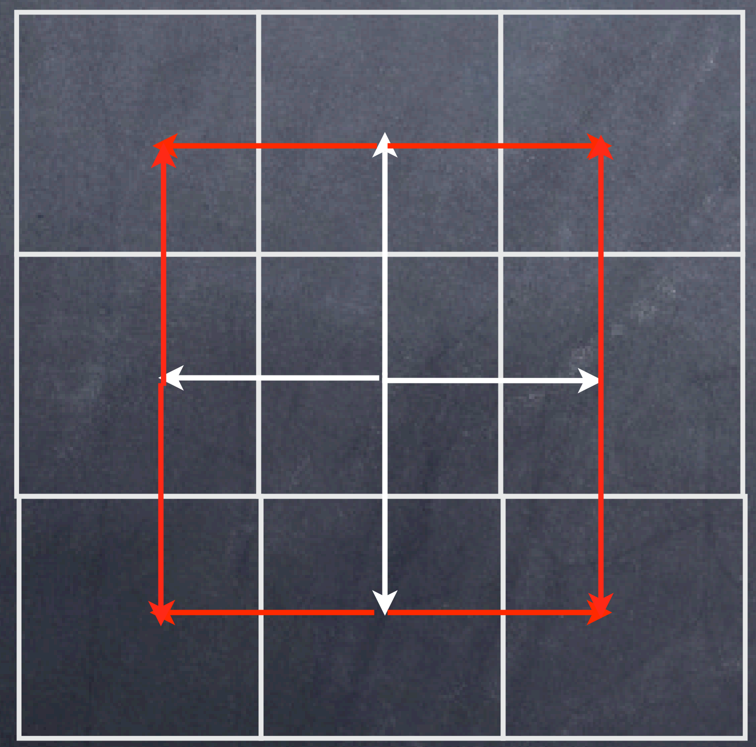

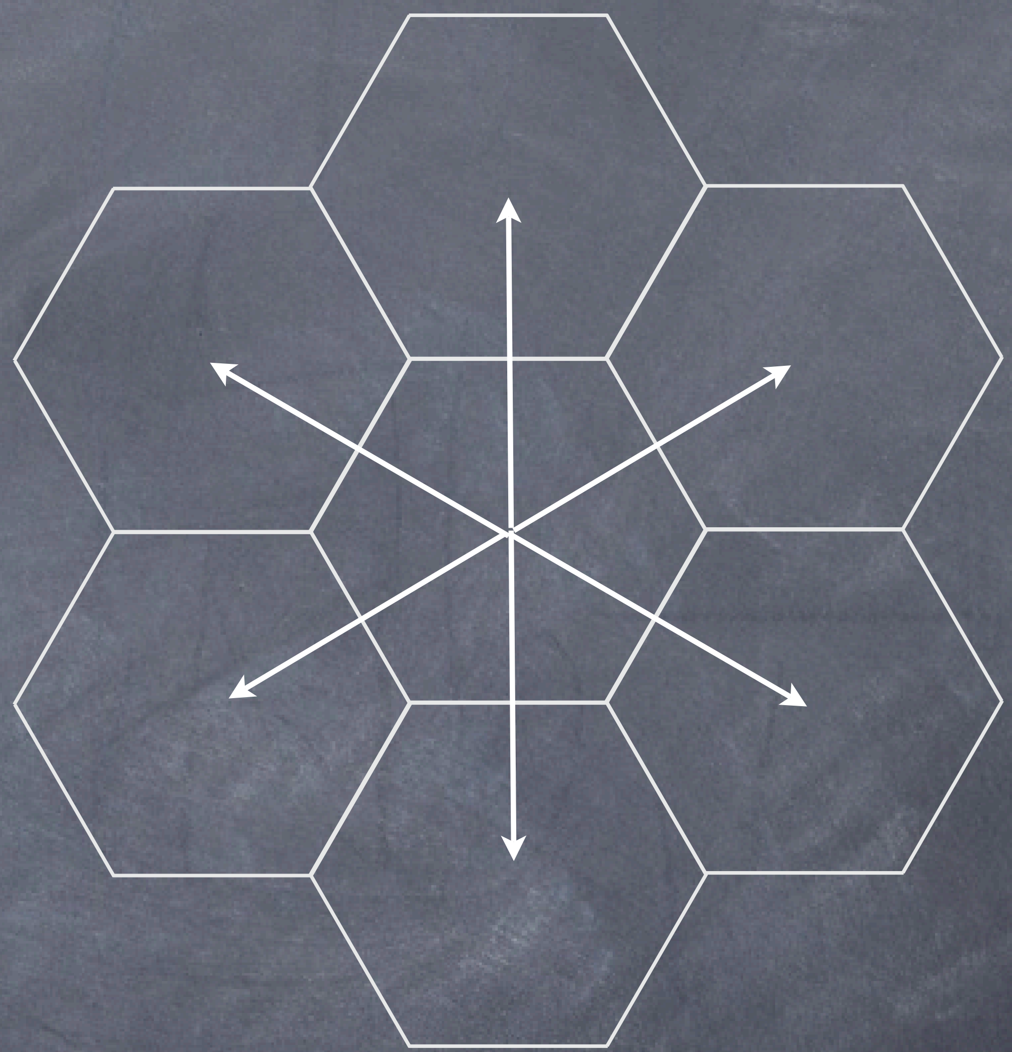

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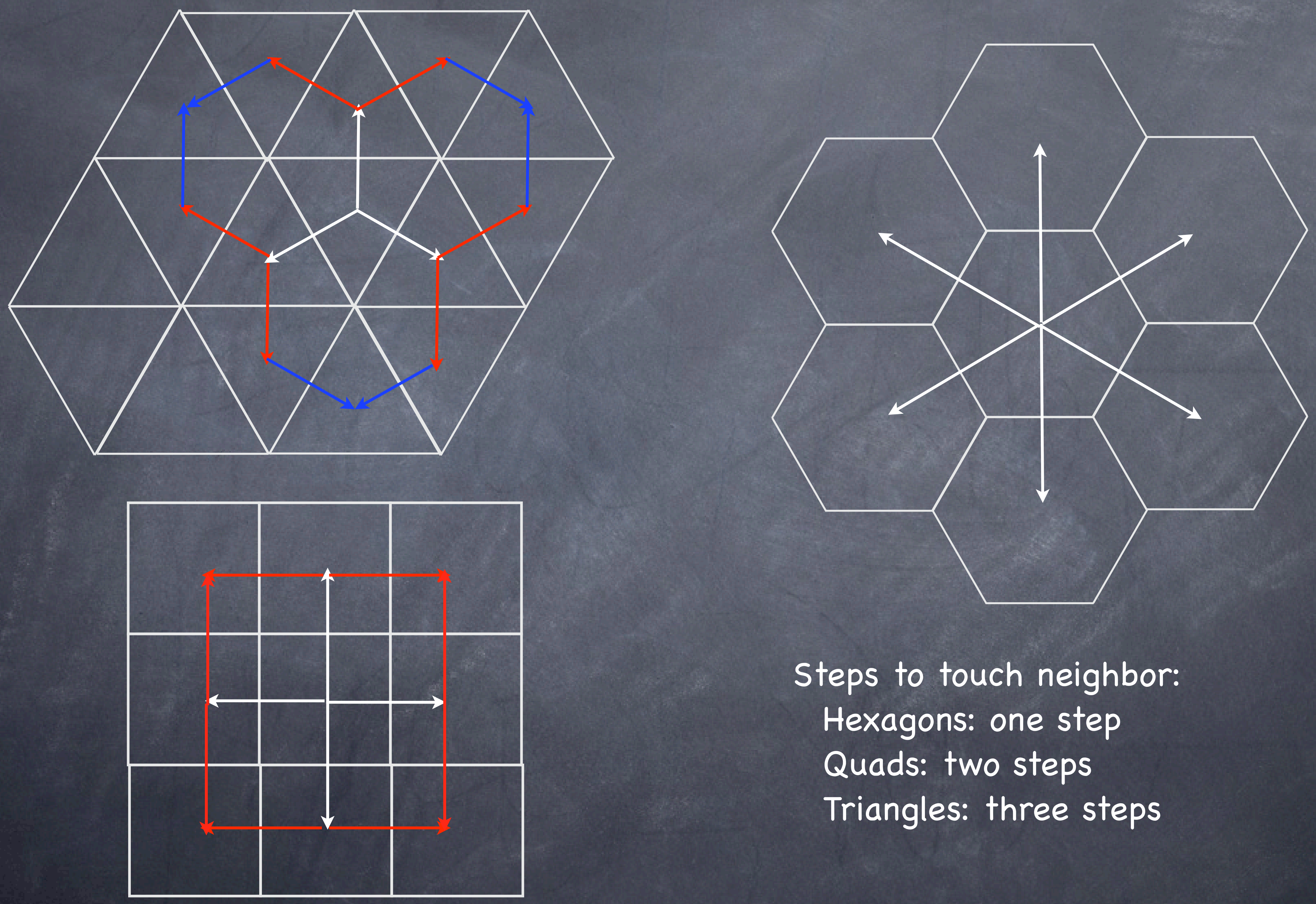


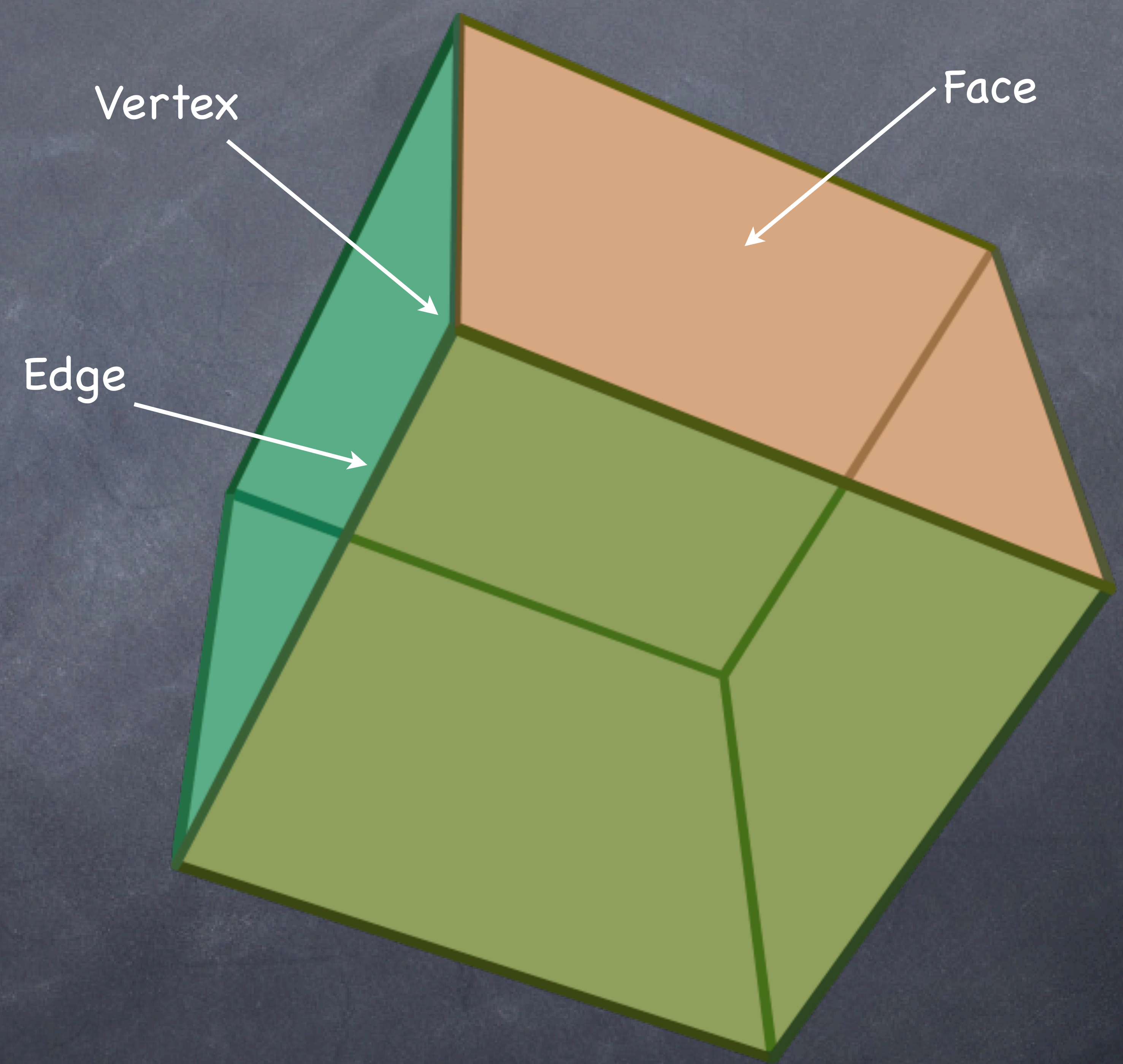
Steps to touch neighbor:

- Hexagons: one step
- Quads: two steps
- Triangles: three steps

In terms of isotropy, the ranking would be (from best to worst): hexagons, quads, triangles.

Mode Counting and Euler's Formula

All convex polyhedra obey the following relationshop: **Faces + Vertices = Edges + 2**
(this formula is related to the angular deficient of 720 degrees)



Mode Counting and Euler's Formula

All convex polyhedra obey the following relationshop: **Faces + Vertices = Edges + 2**
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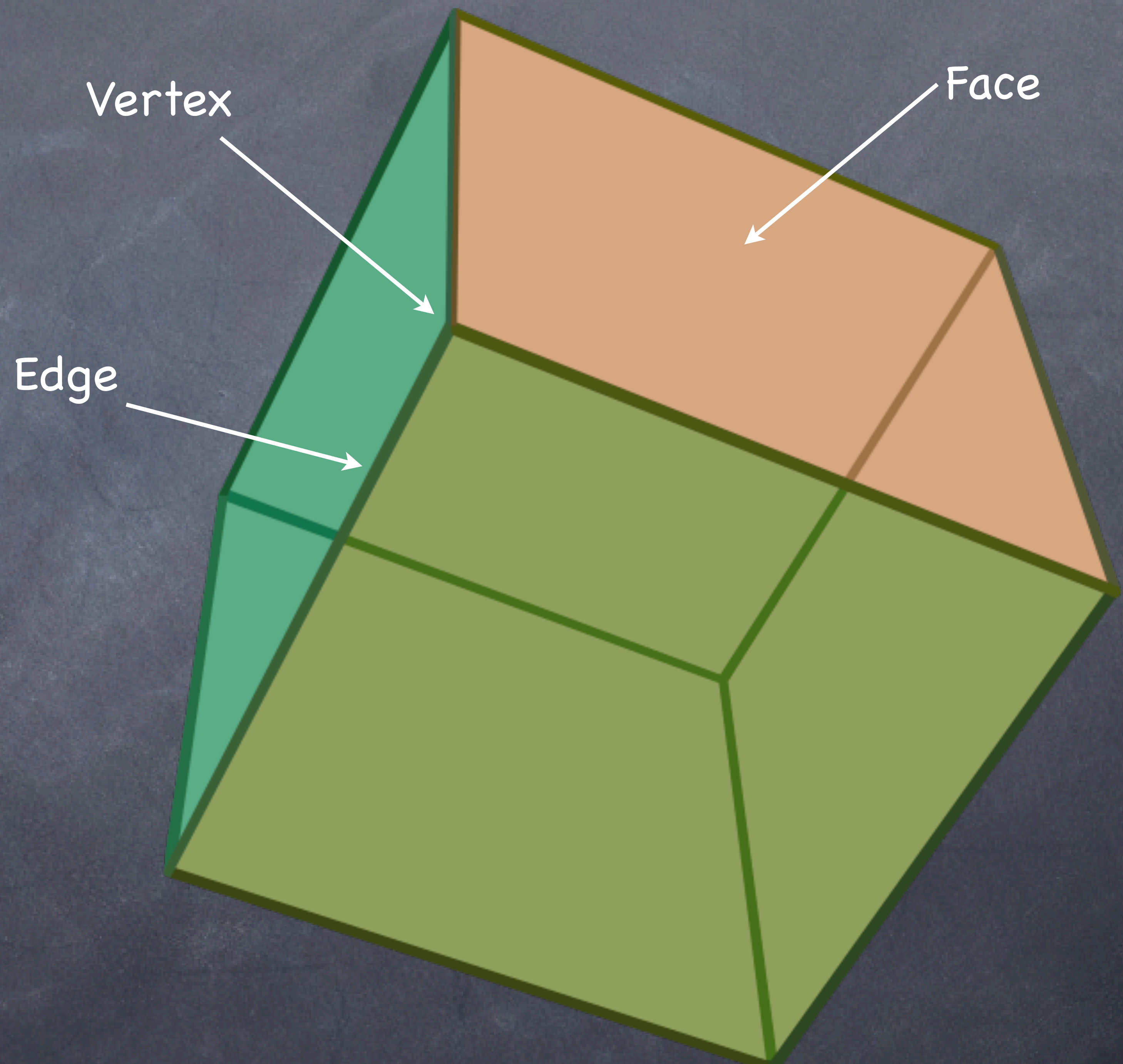
Faces = 6
Vertices = 8
Edges = 12

6 + 8 = 12 + 2

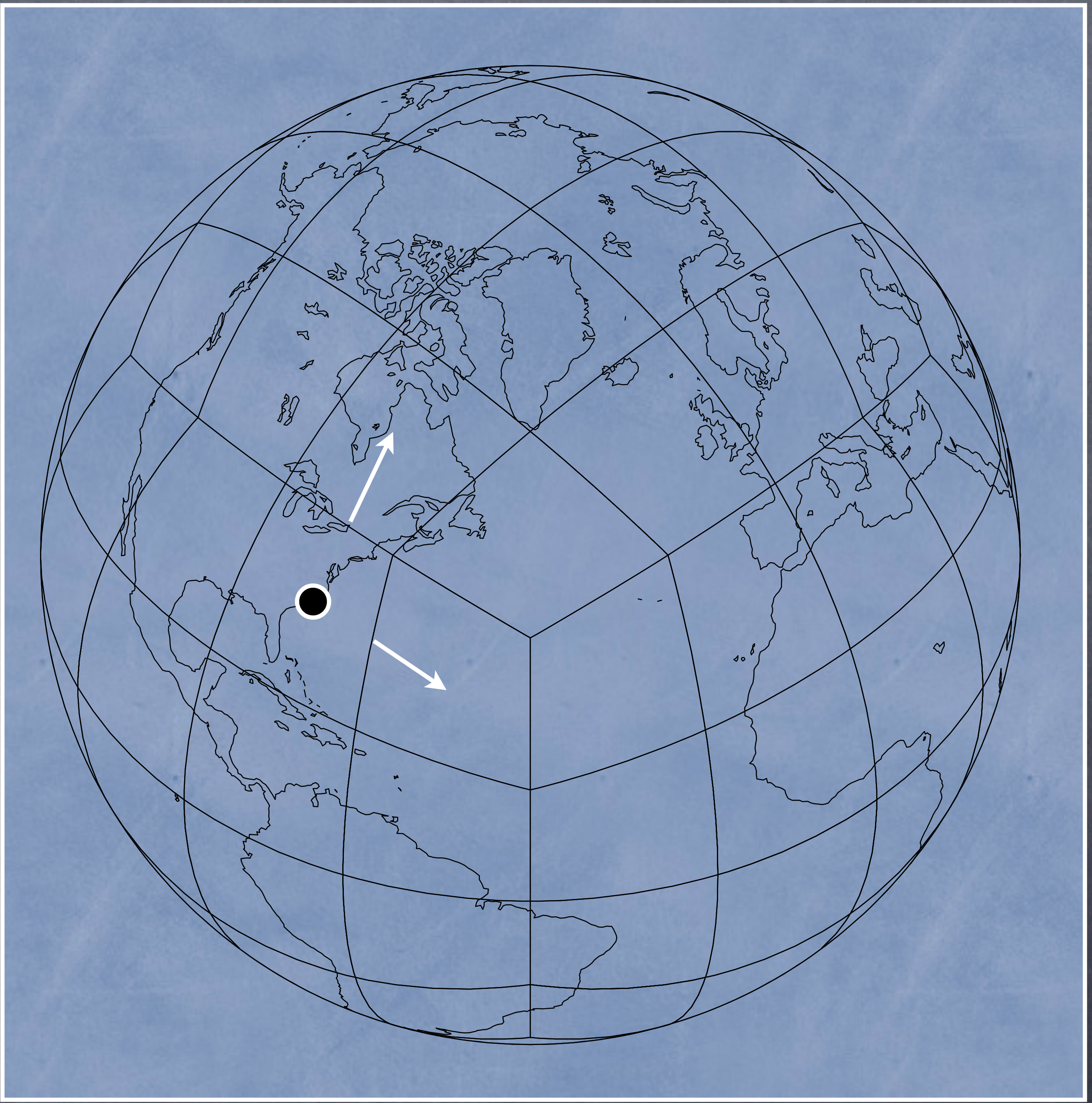


C-grid staggering: every edges owns one velocity component.

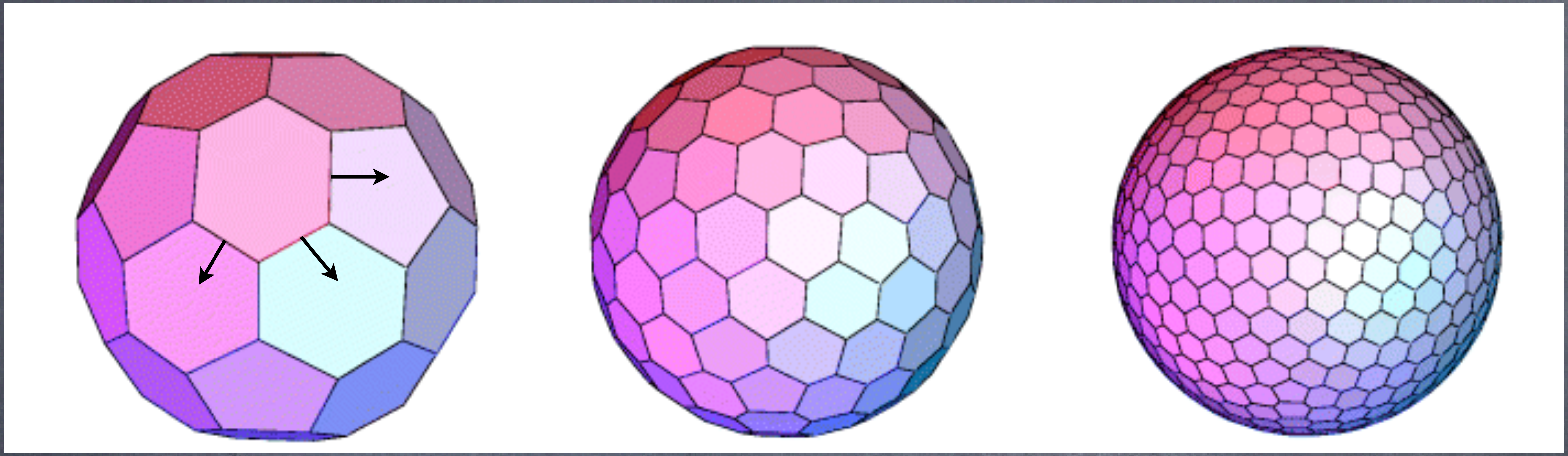
Faces = 96
Vertices = 98
Edges = 192

96 + 98 = 192 + 2

We have exactly twice the number of edges as faces, so the mode counting works out: two velocity degrees of freedom for each mass degree of freedom.



C-grid staggering: every edges owns one velocity component. It is not so clean on the icosahedron ....



Faces = 642
Vertices = 1280
Edges = 1920

642 + 1280 = 1920 + 2

For every face there are approximately three edges, so there are three velocity components for every one mass point. One way to look at this is that the velocity field is not fully constrained by the mass field, i.e. the velocity field is under-determined.

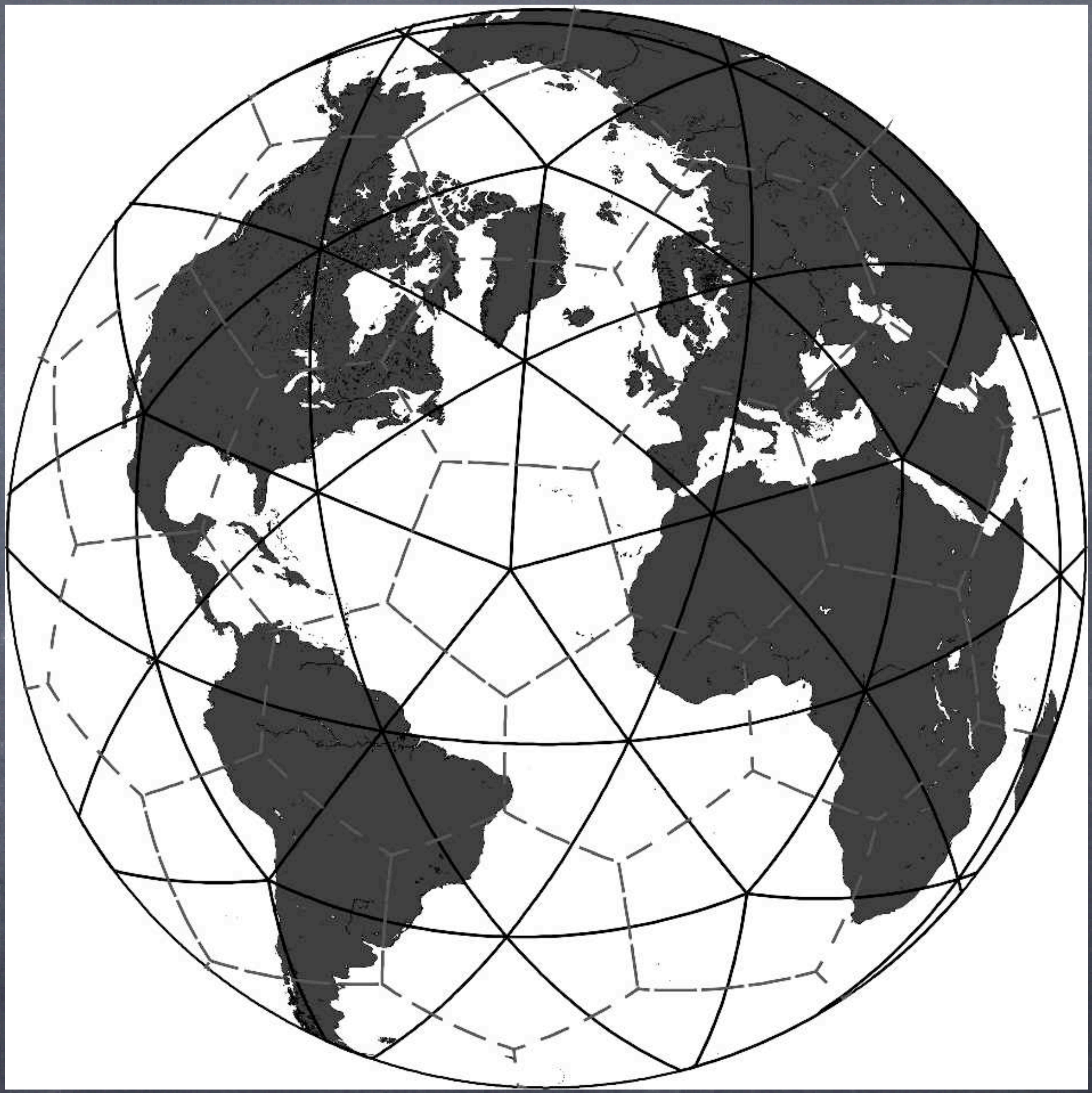
We will revisit this issue later.

C-grid staggering: every edges owns one velocity component. It is not so clean on the triangles either ....

Faces = 1280
Vertices = 642
Edges = 1920

1280 + 642 = 1920 + 2

Now there are only about 1.5 velocity components per mass point. In this case, the mass field is under-constrained.



So in terms of having the appropriate match between mass points and velocity points, the ranking (from best to worst) is: quad, hexagons and triangles. (I put hexagons ahead of triangles because I think extra modes in the velocity field are easier to deal with than extra modes in the mass field).

Having a mismatch between mass and velocity modes is not (necessarily) a show-stopper. But, in general, the field that is under-constrained will be susceptible to noise. Take must be taken.

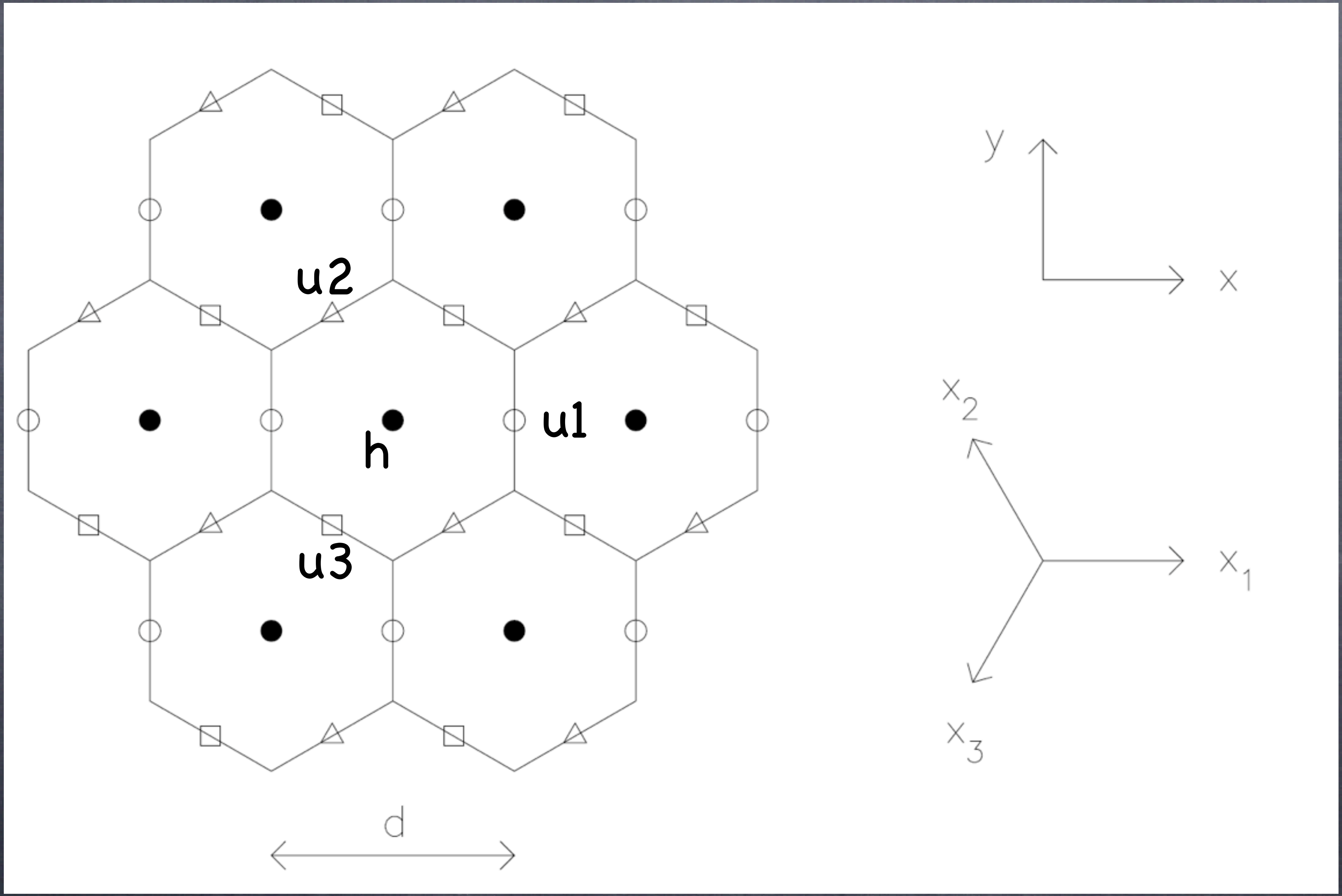
Dispersion relations on hexagons or triangles ...

Dispersion relations on hexagons or triangles ...

Determining dispersion relations on grids composed of something other than squares is not trivial:

1) The grids are not tensor products
2) Various (k,l) combinations can have the same representation on grid.
3) The resolved wavenumber space is not obvious.
4) (Triangles): nodes are not "evenly" distributed.

System definition for hexagons.



Discrete linearized equations: C-grid hexagons

$$\partial_t \Phi + \frac{2}{3}\Phi_0 \left(\delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3\right) = 0,$$

$$\partial_t u_1 - \frac{f_0}{\sqrt{3}}\left(\overline{u_2}^3 - \overline{u_3}^2\right) + \delta_1 \Phi = 0,$$

$$\partial_t u_2 - \frac{f_0}{\sqrt{3}}\left(\overline{u_3}^1 - \overline{u_1}^3\right) + \delta_2 \Phi = 0,$$

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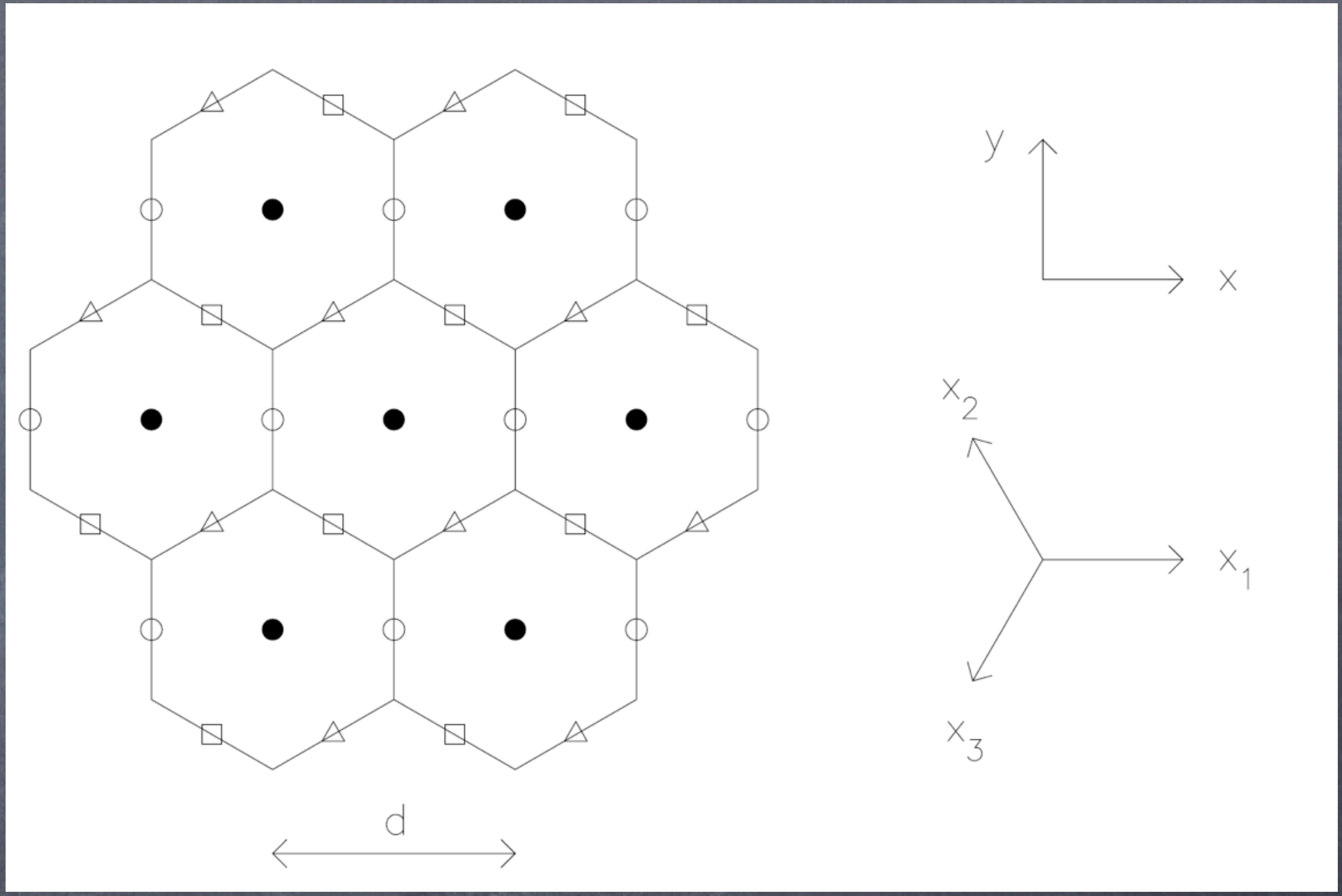

Linearized about state of rest (u=0, h=constant)

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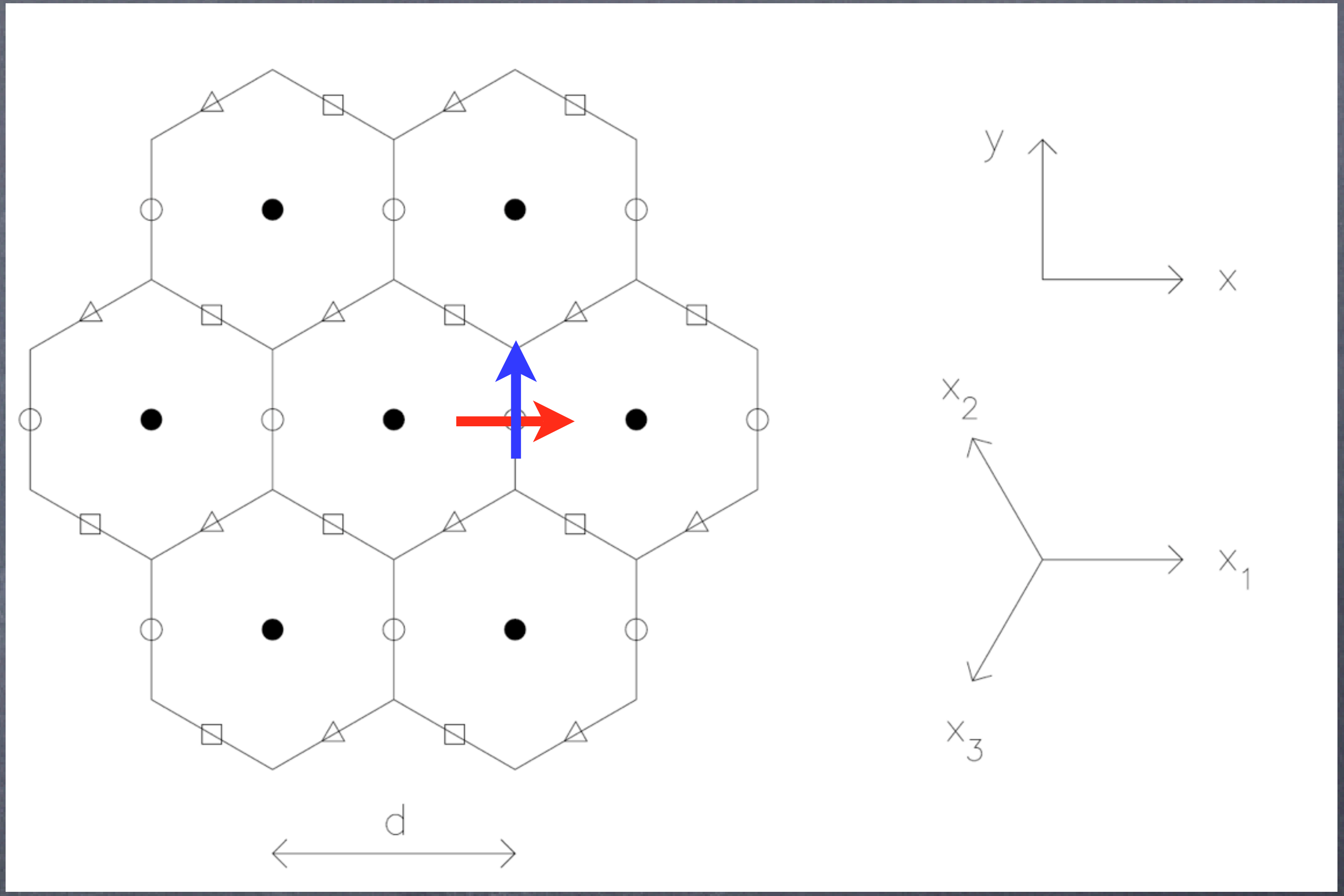

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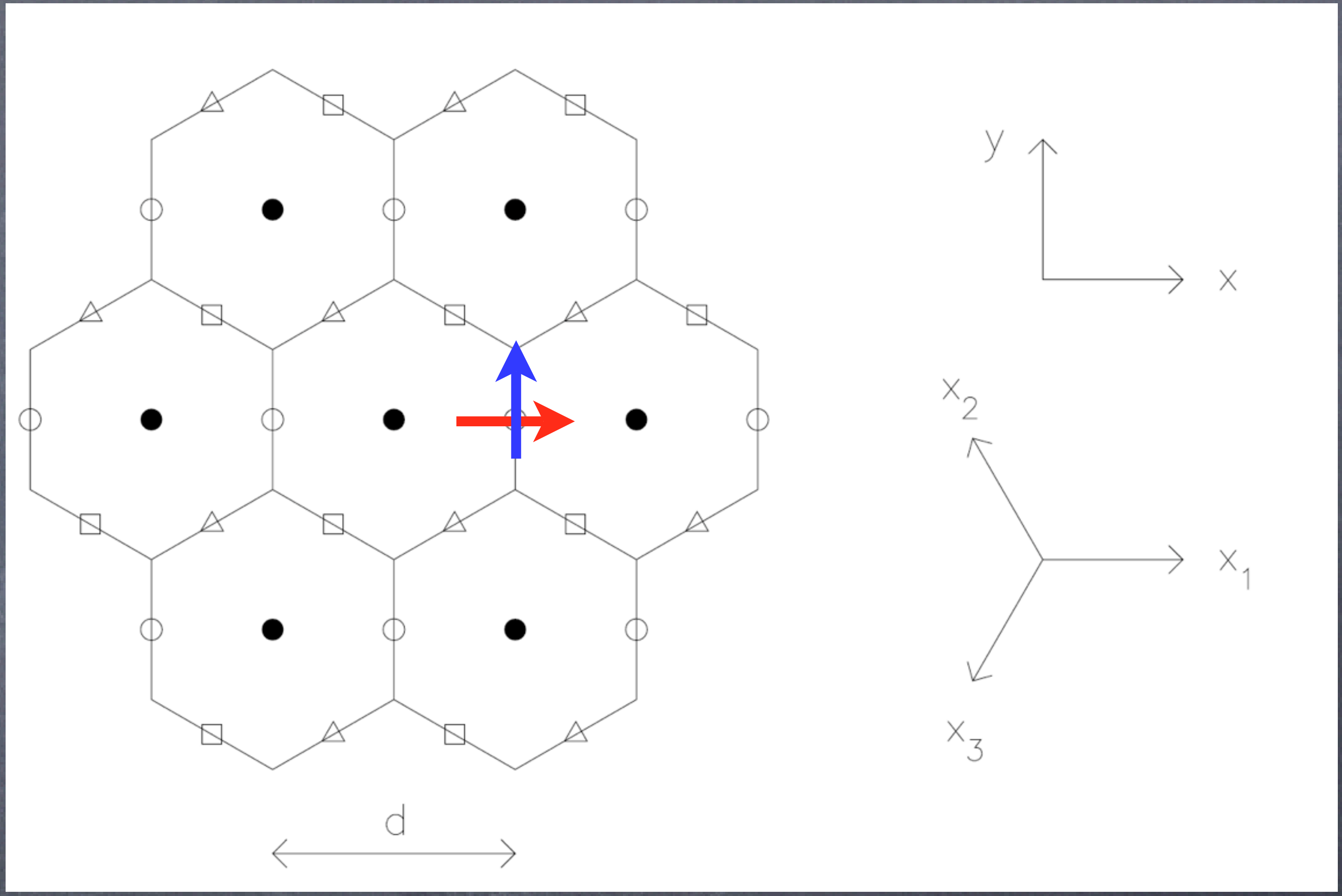
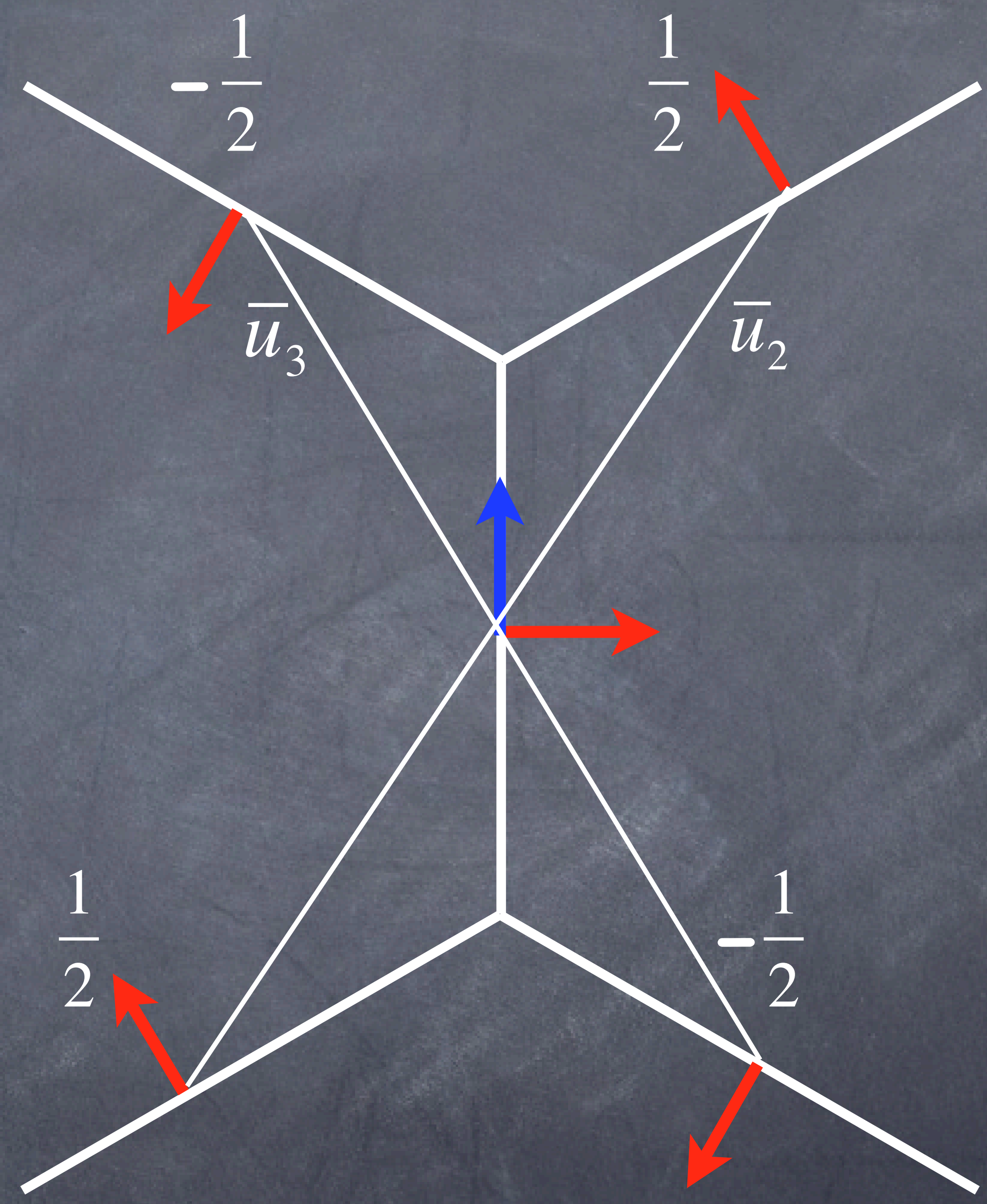


Linearized about state of rest (u=0, h=constant)

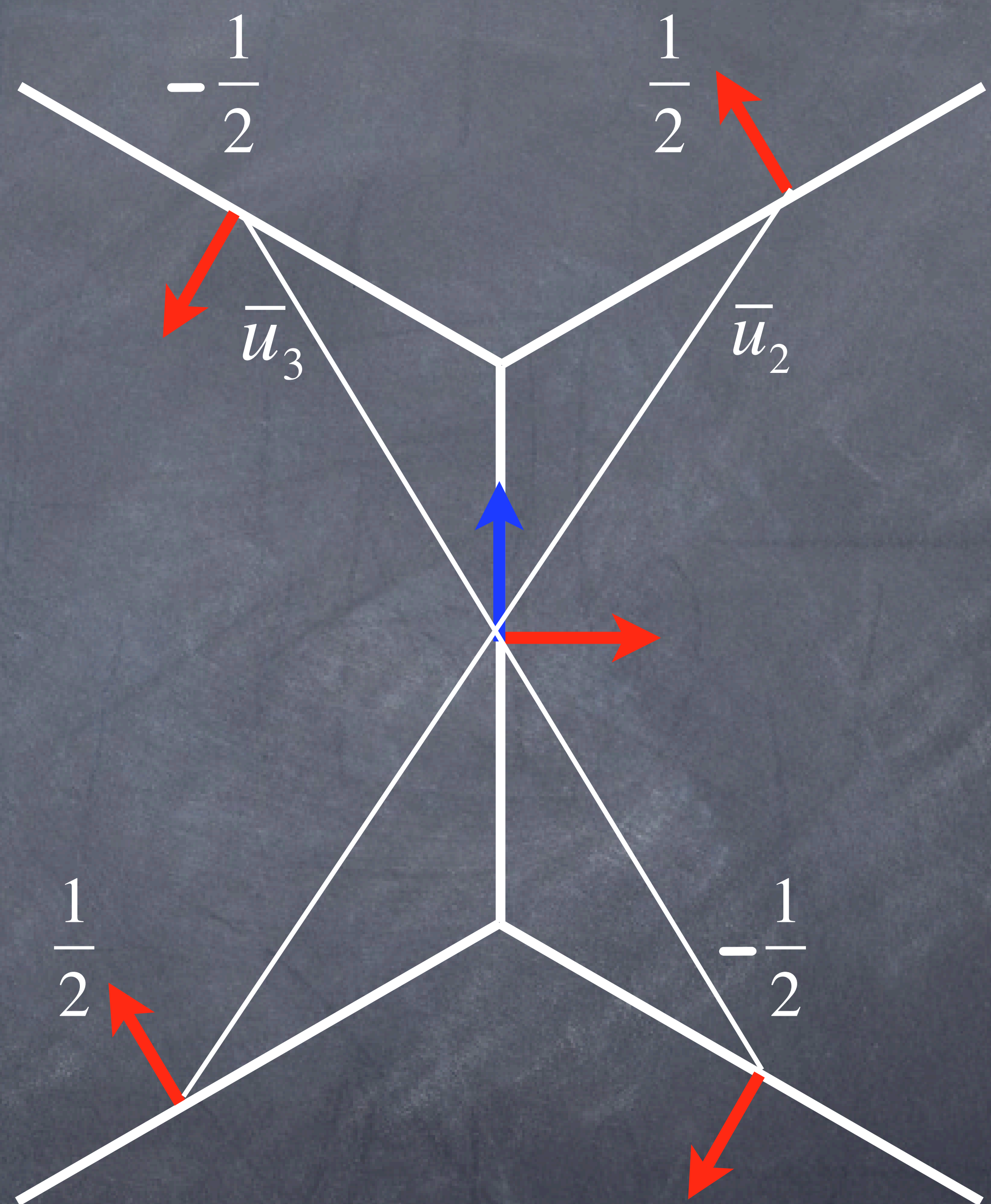
Coriolis Force Averaging: Hex C-grid (unmod)



Coriolis Force Averaging: Hex C-grid (unmod)

Continuous solution:

$$\frac{\omega}{\mathrm{f}} = 0$$:geostropic balance



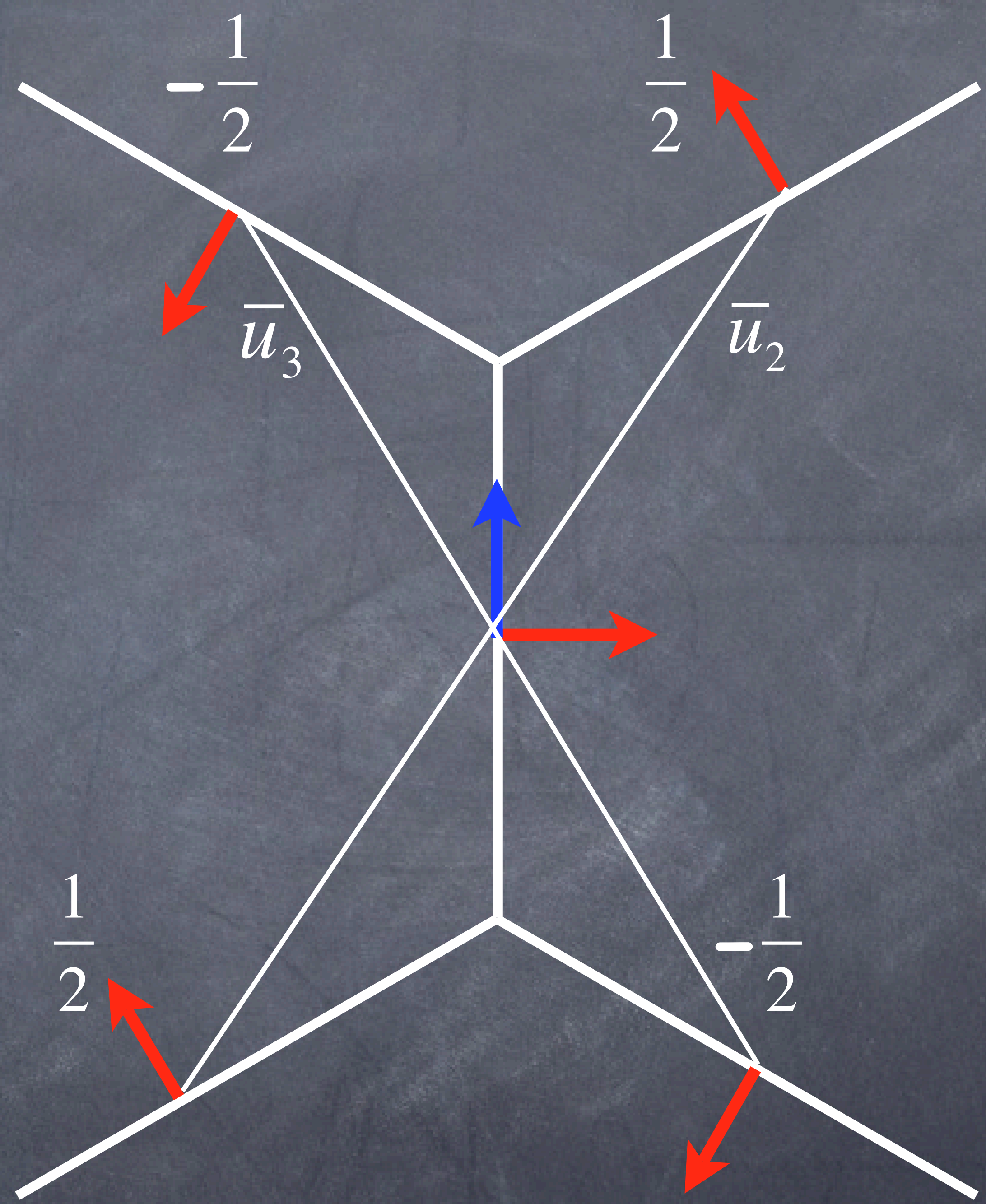
Coriolis Force Averaging: Hex C-grid (unmod)

Continuous solution:

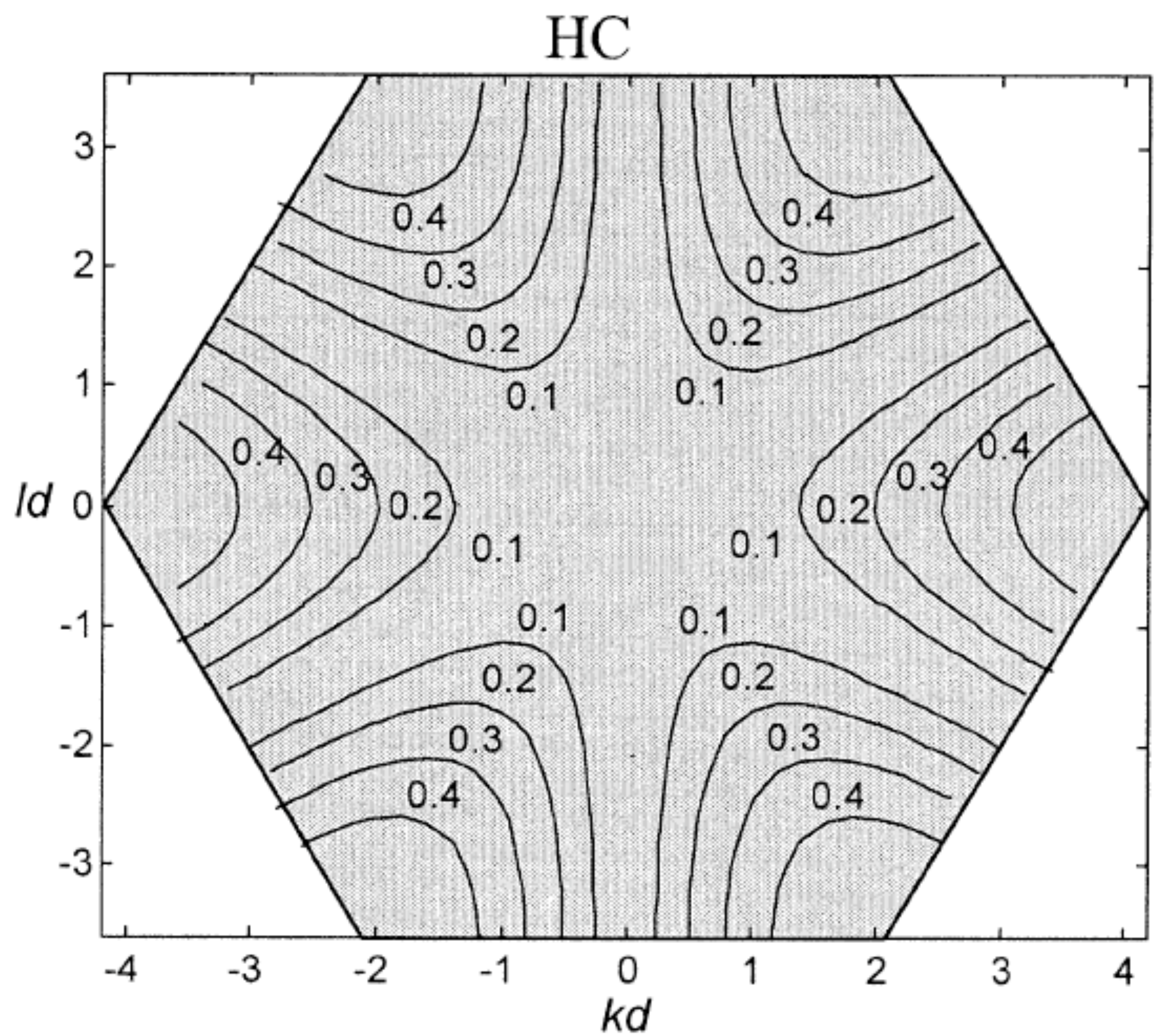
$$\frac{\omega}{f} = 0$$:geostropic balance

Discrete solution:

$$\frac{\omega}{f} = \sqrt{B} \neq 0$$

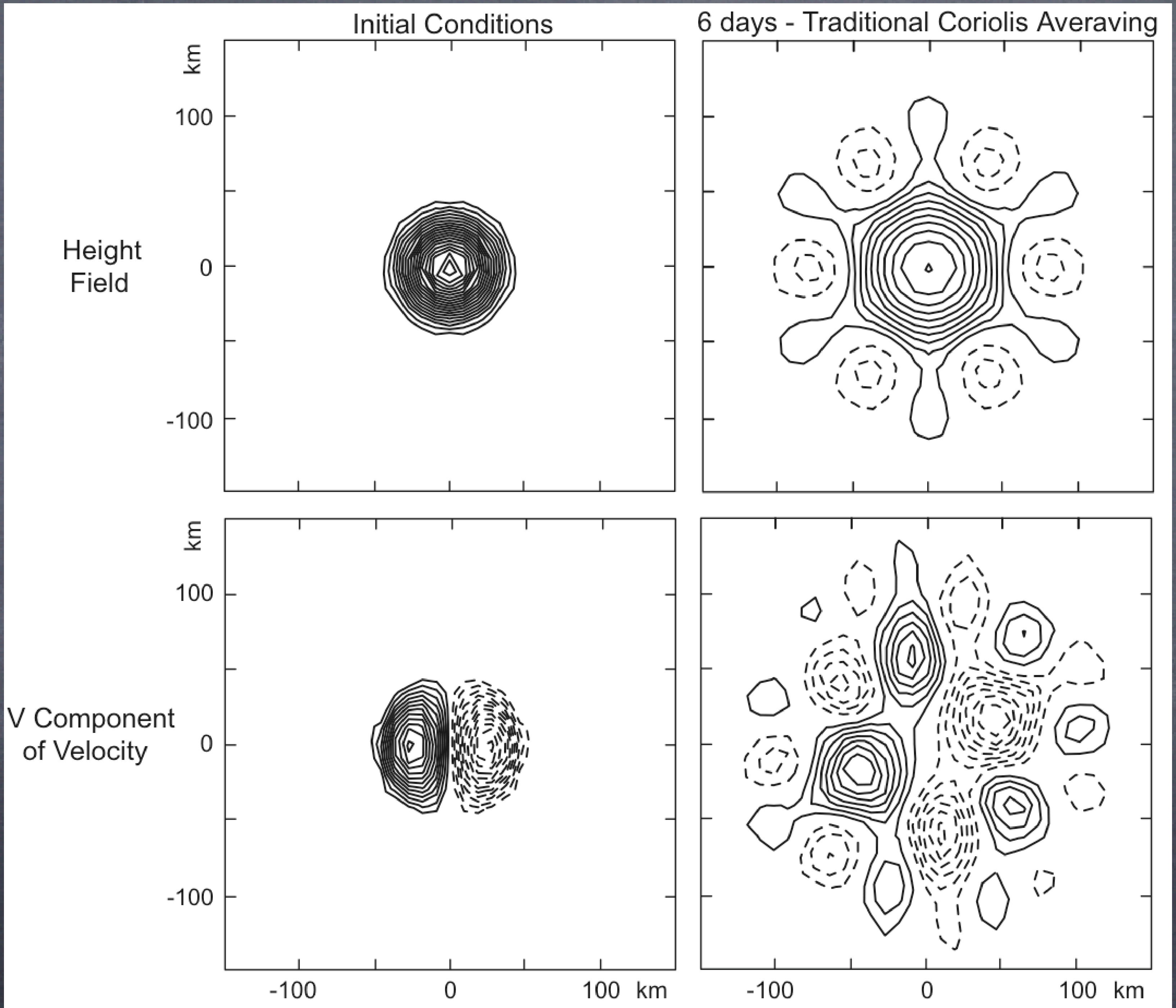


Non-zero geostrophic mode



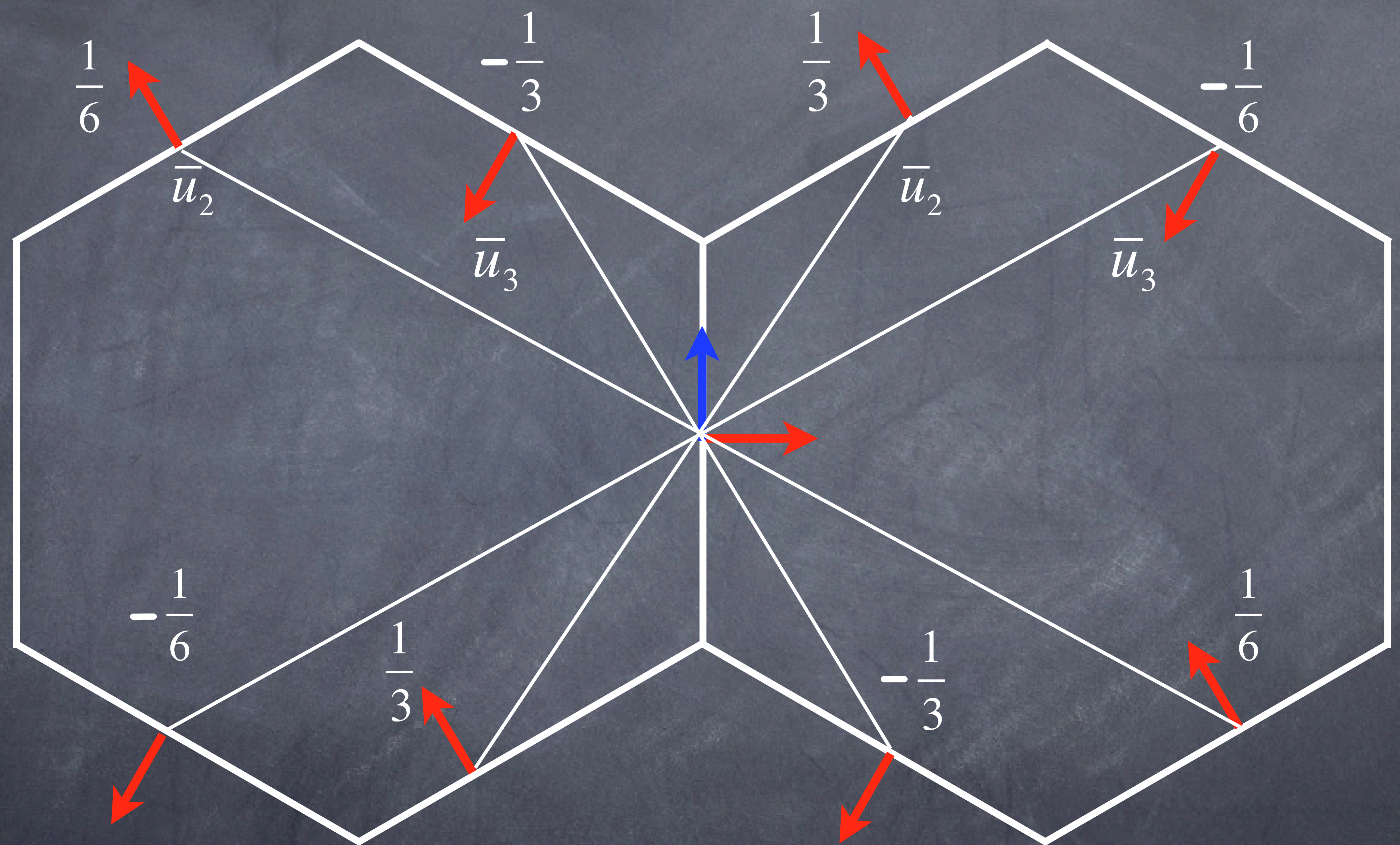
Nickovic et al 2002

Implications of a non-zero geostrophic mode...



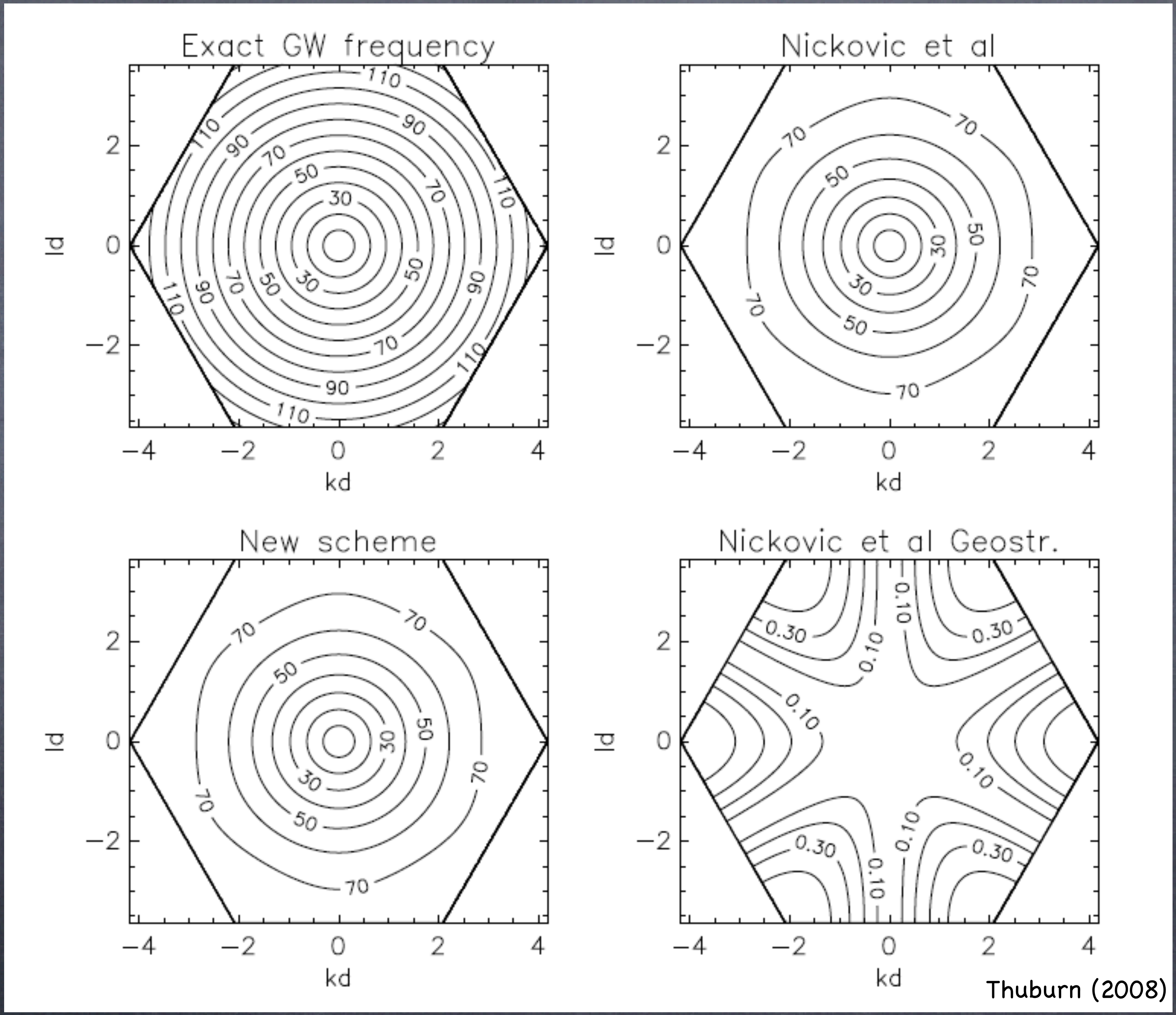
courtesy of Joe Klemp

Alternative Averaging Methods ....

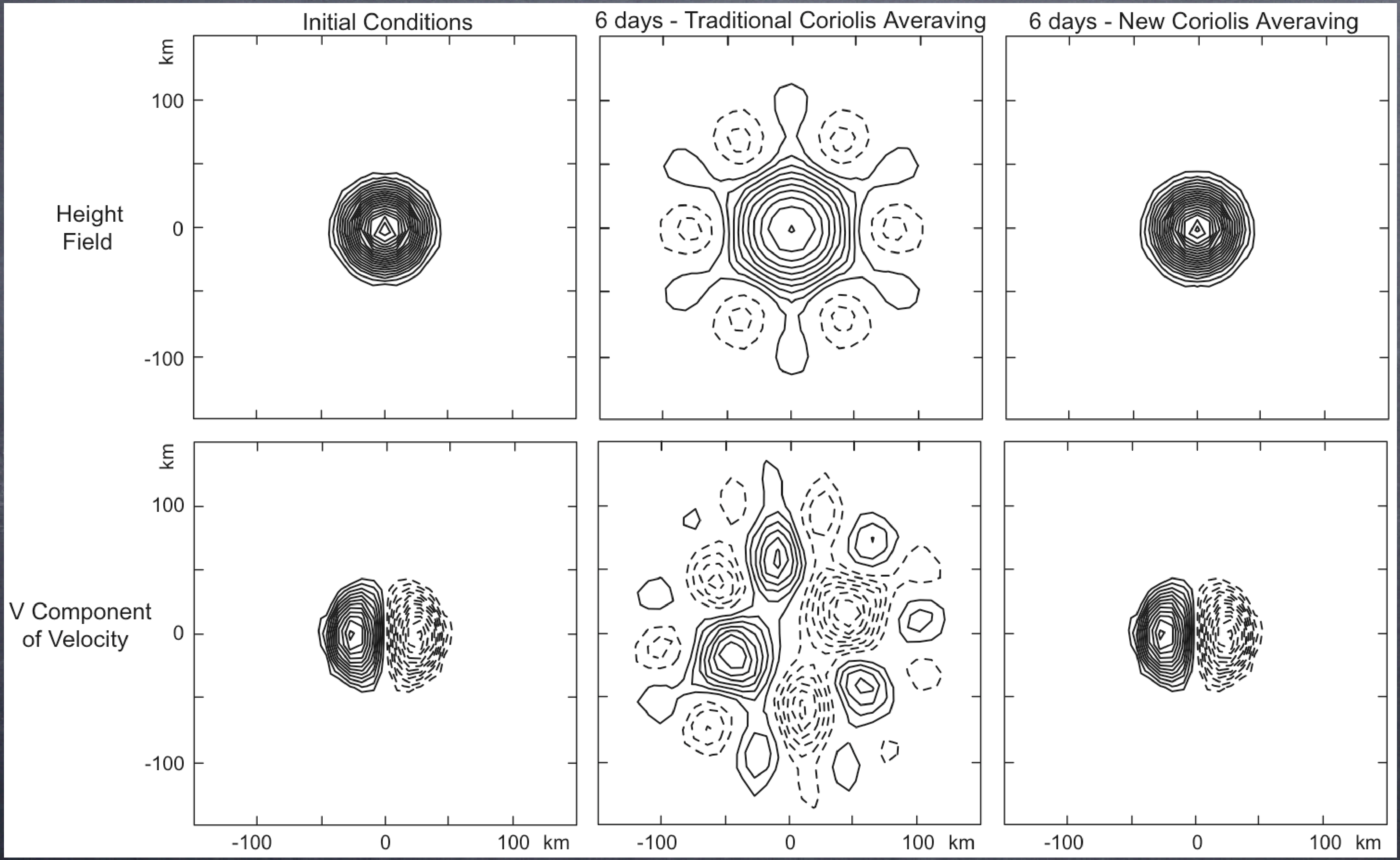


method independently derived by Klemp and Skamarock (2008) and Thuburn (2008).

This recovers geostrophic balance and produces an acceptable gravity wave (GW) dispersion relations.



Geostrophic adjustment with new averaging.



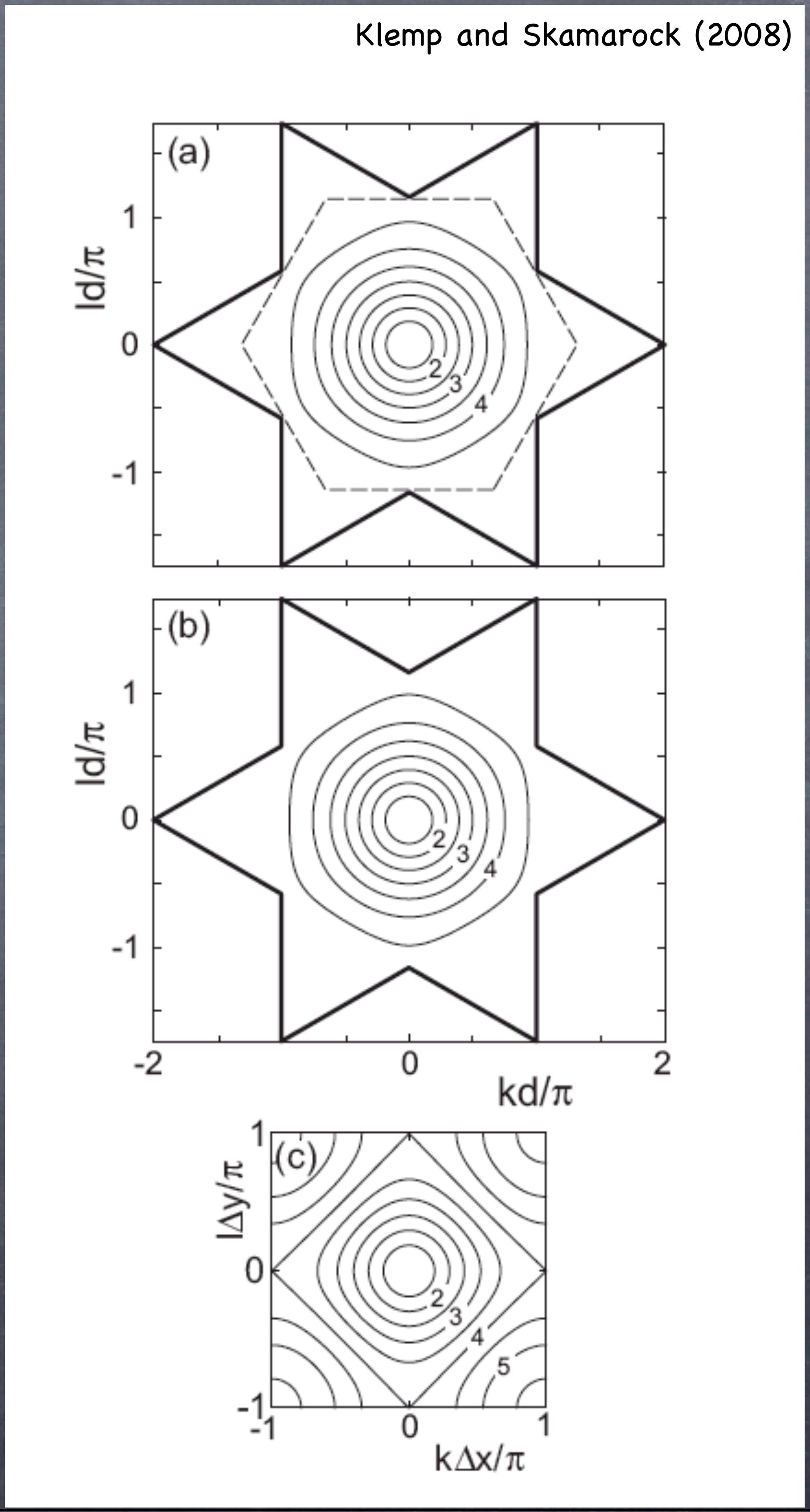
courtesy of Joe Klemp

Comparison of gravity wave modes for hexagons and squares ....

old averaging ...

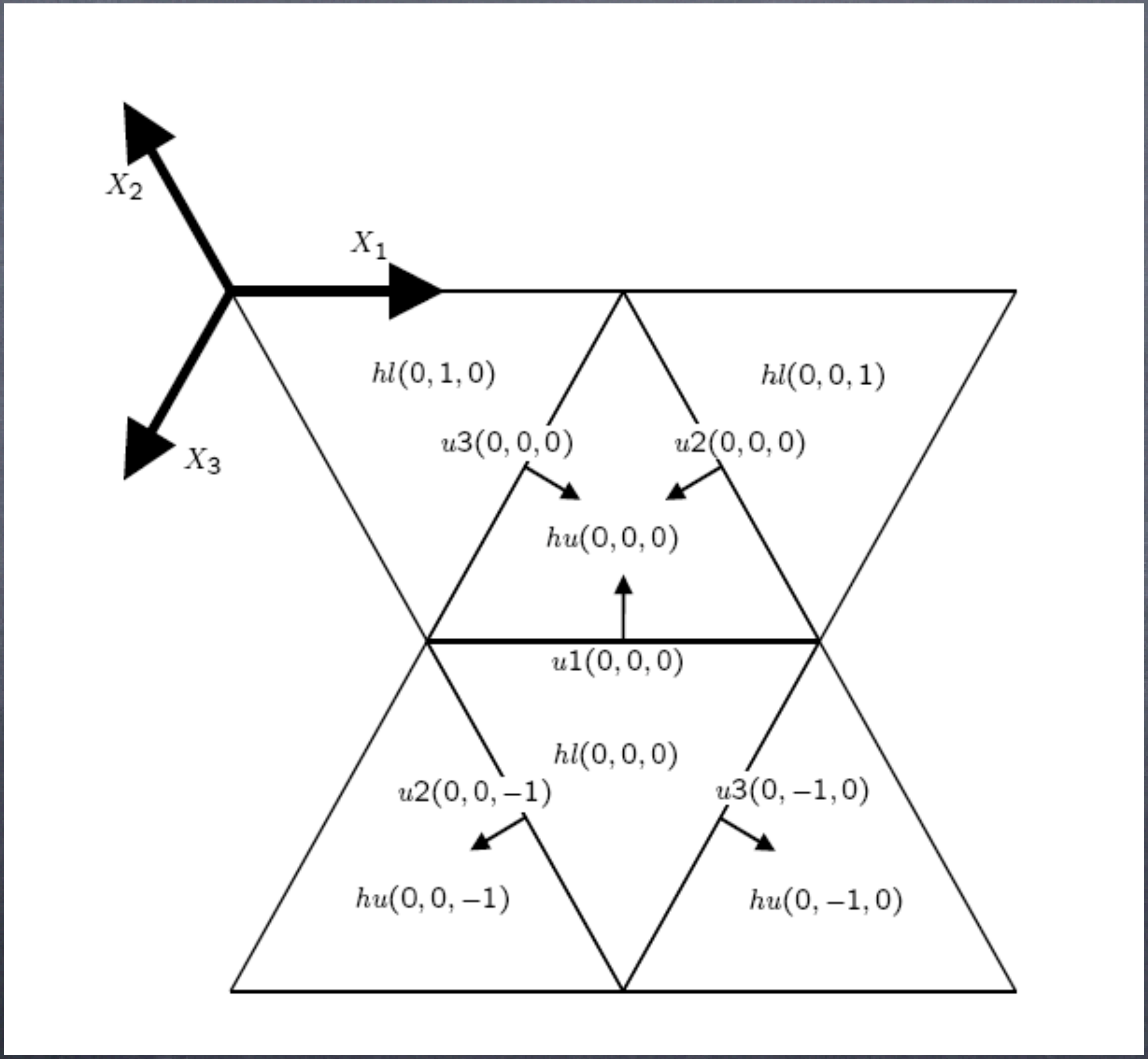
new averaging ...

squares (domain scaled to match degrees of freedom on hex grid).



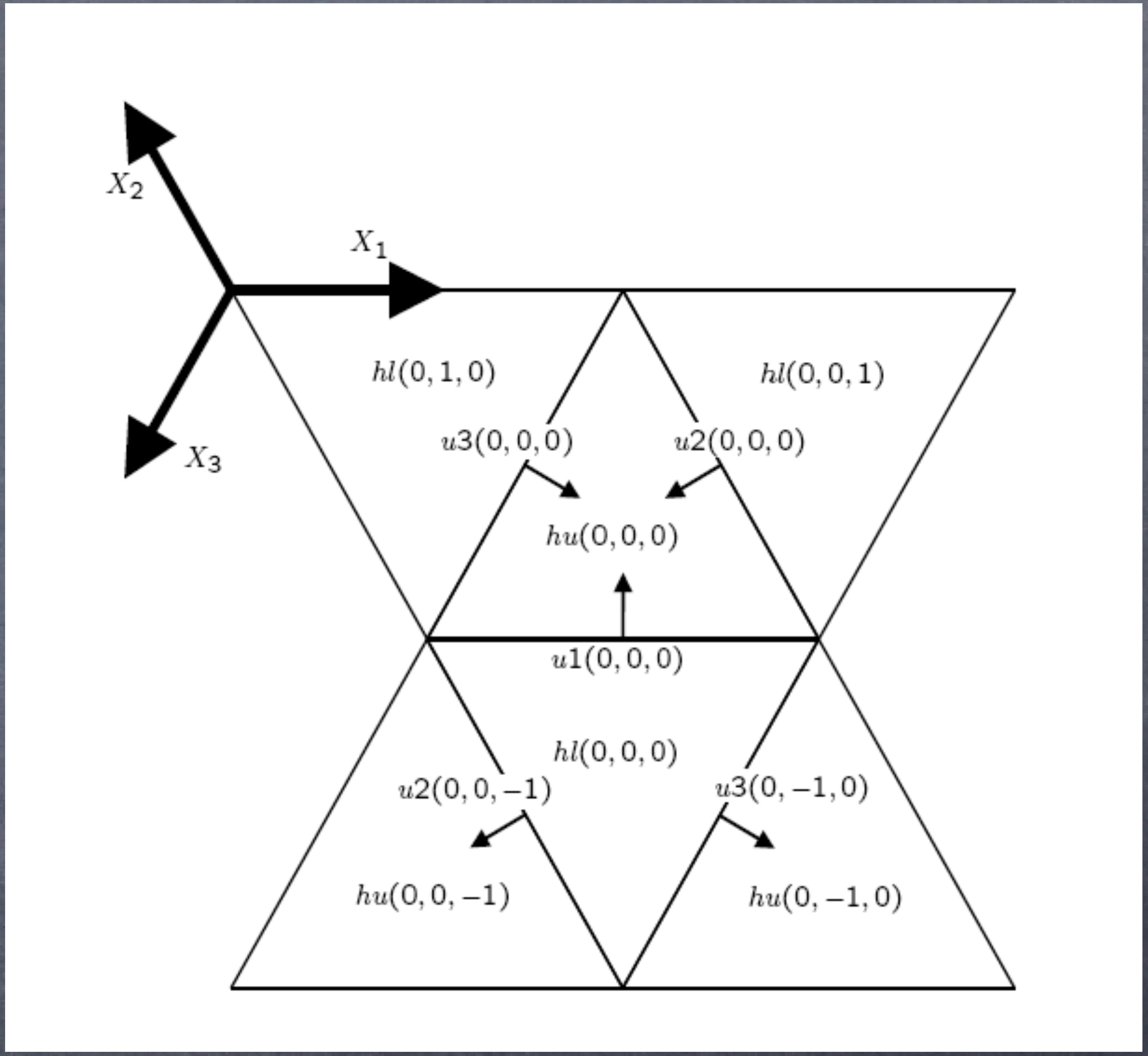
Dispersion relation on triangles ....

dispersion analysis on triangles is significantly more complicated than on hexagons, because there are really two types of triangles: those pointing up and those pointing down. (ongoing work: Bonaventura, Klemp and Ringler)

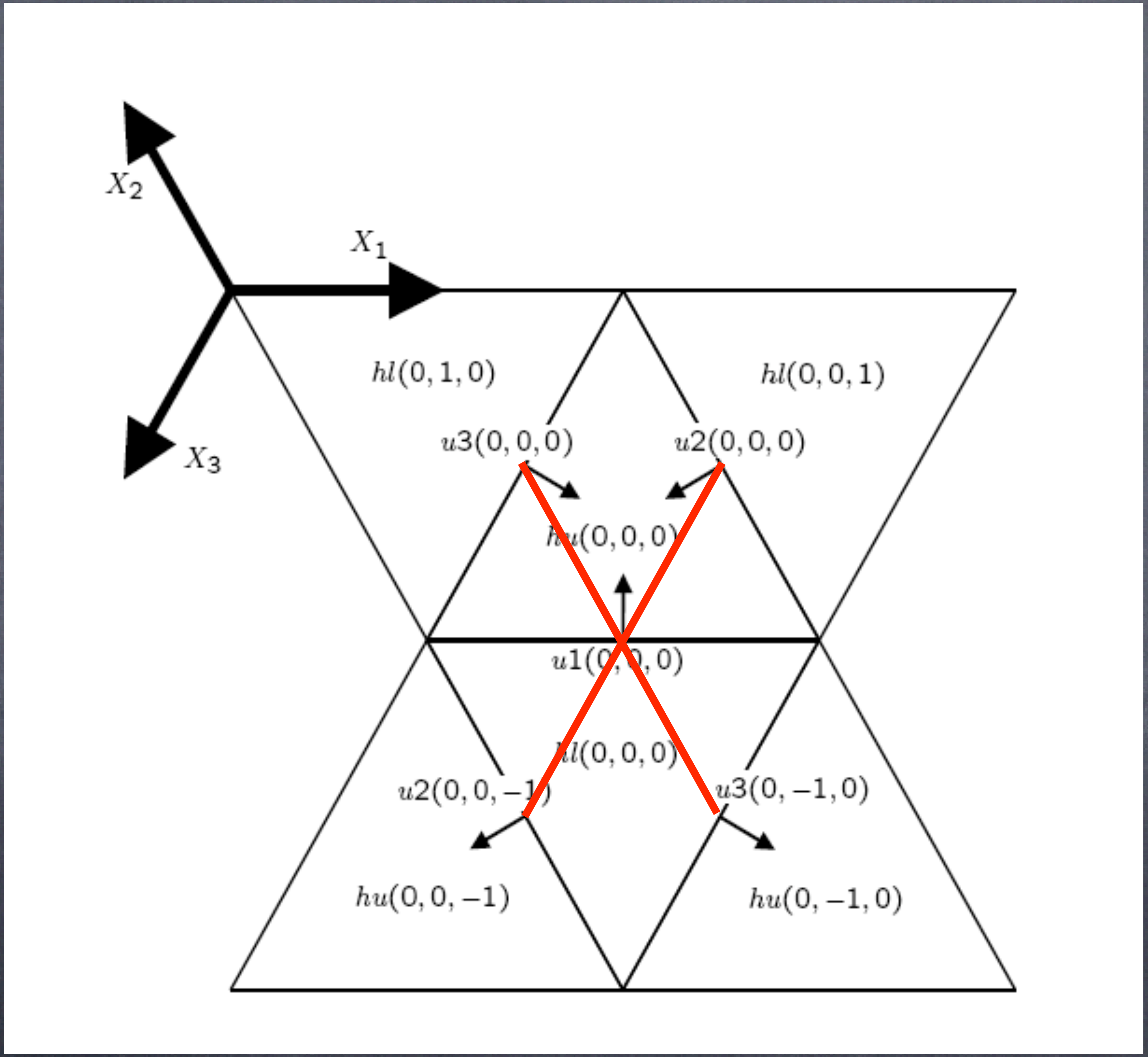


System definition on triangles ....

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= -\frac{f}{\sqrt{3}}(\bar{u}_3^2 - \bar{u}_2^3) - \frac{g}{d}(h_u - h_l) \\
\frac{\partial u_2}{\partial t} &= -\frac{f}{\sqrt{3}}(\bar{u}_1^3 - \bar{u}_3^1) - \frac{g}{d}(h_u - h_l) \\
\frac{\partial u_3}{\partial t} &= -\frac{f}{\sqrt{3}}(\bar{u}_2^1 - \bar{u}_1^2) - \frac{g}{d}(h_u - h_l) \\
\frac{\partial h_u}{\partial t} &= \frac{4H}{3d}(u_1 + u_2 + u_3) \\
\frac{\partial h_l}{\partial t} &= -\frac{4H}{3d}(u_1 + u_2 + u_3).
\end{aligned}
$$

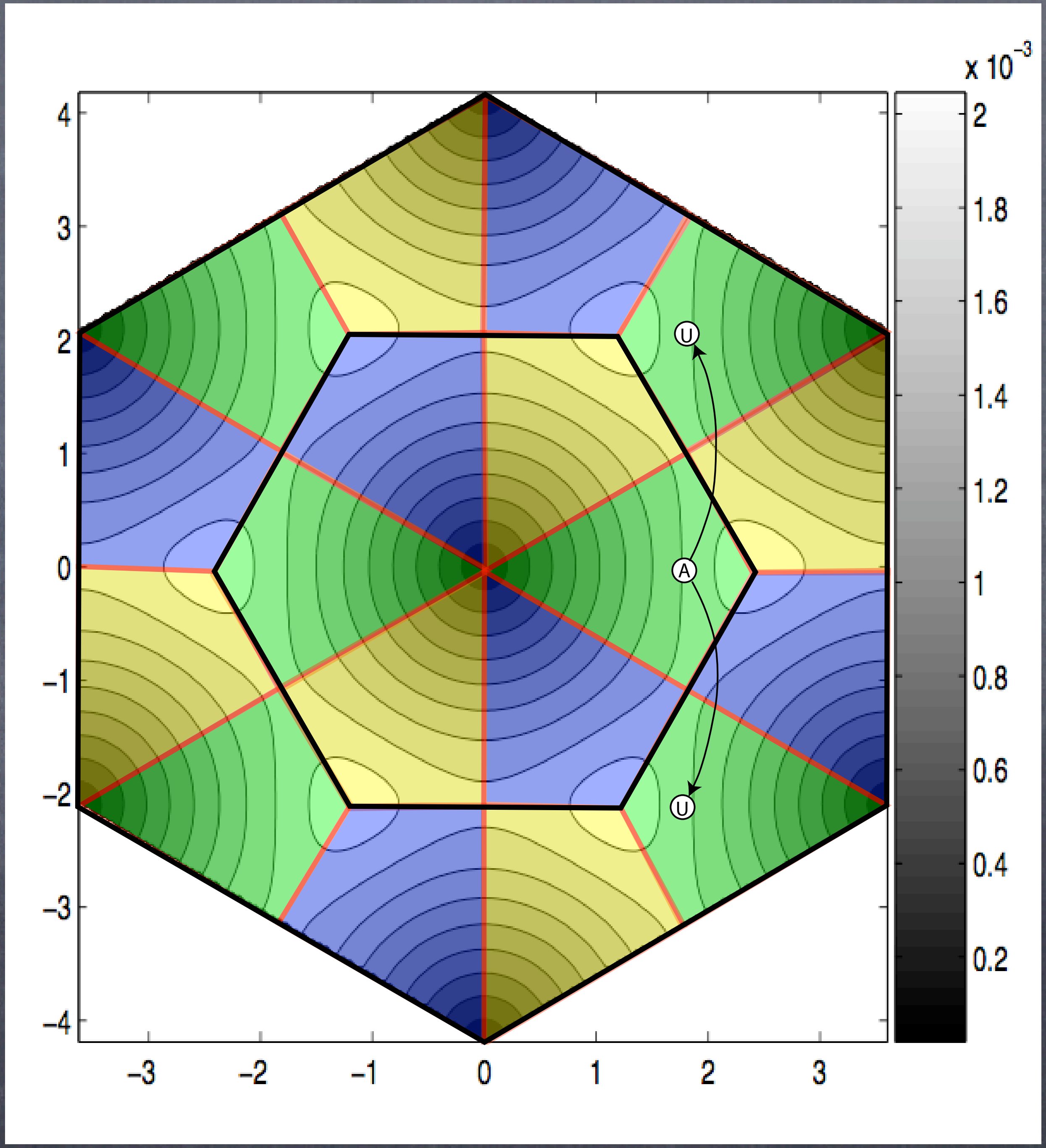


Oddly enough, the Coriolis averaging that caused problems on the hexagonal grid is benign on the triangular grid ....



Dispersion relation on triangular C-grid.

Overall, the dispersion relation is not as uniform as on the hexagonal grid. (This is consistent with our isotropy discussion).



Dispersion relation on triangular C-grid.

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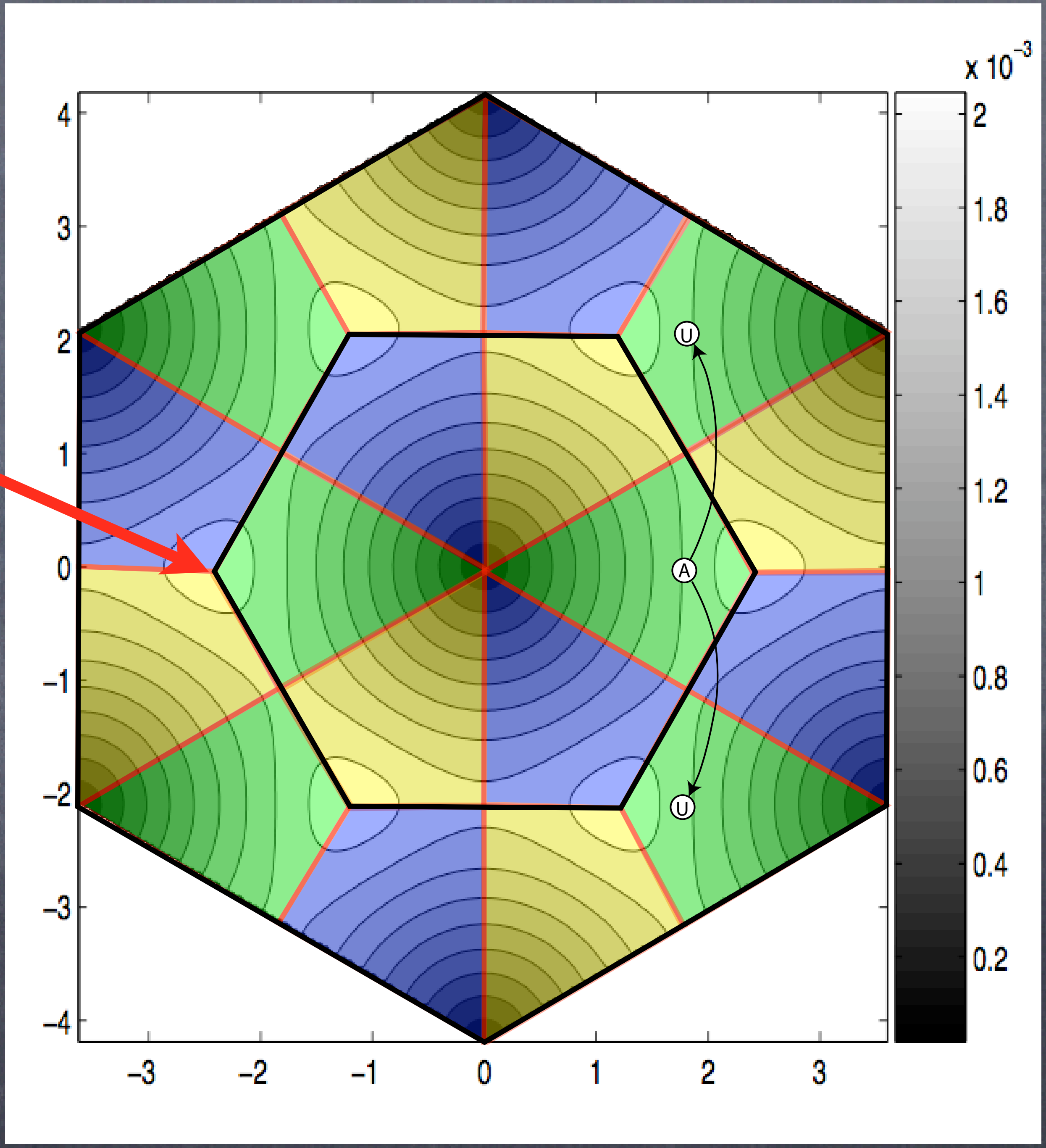
zero group velocity locations

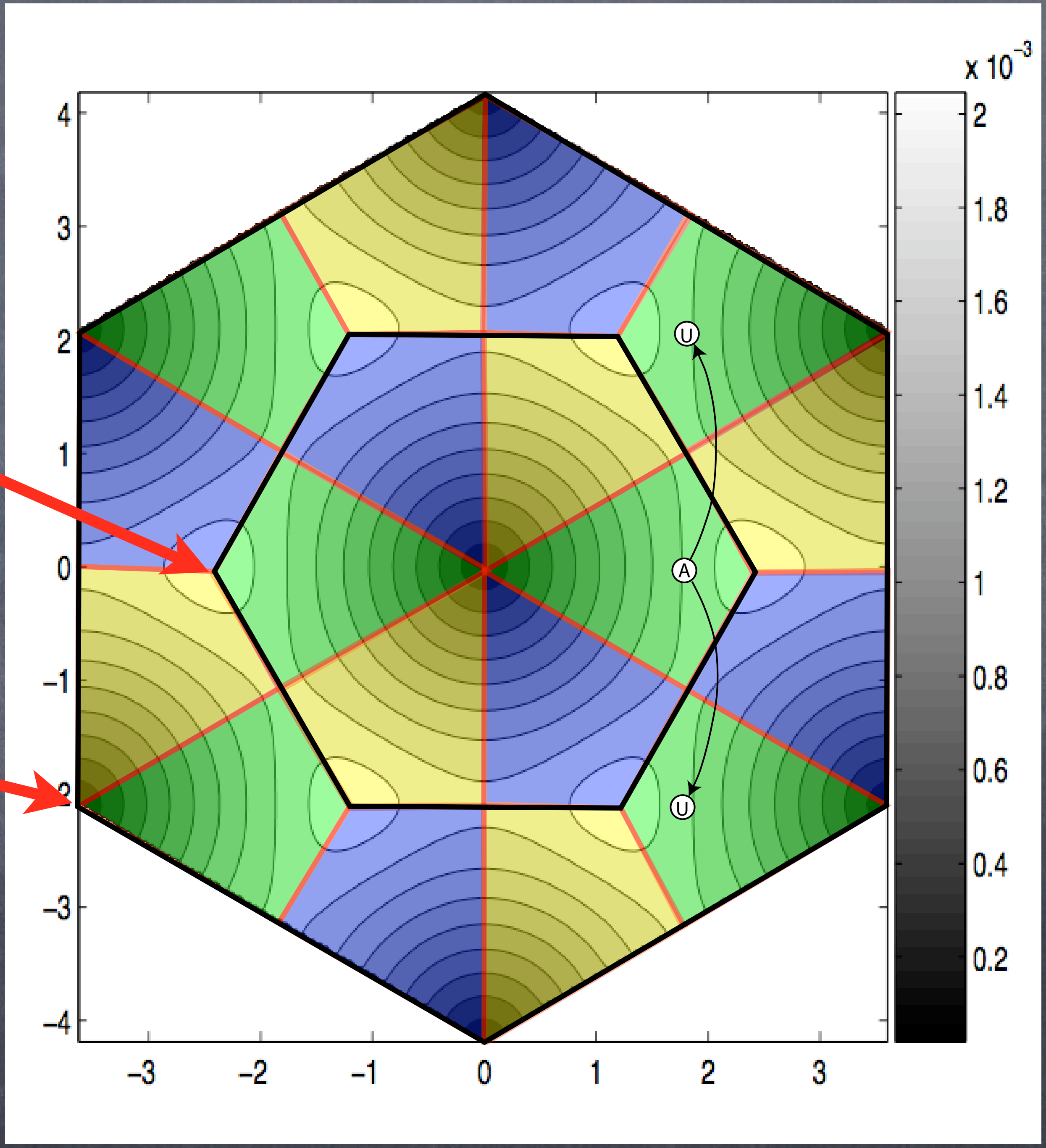


Dispersion relation on triangular C-grid.

Overall, the dispersion relation is not as uniform as on the hexagonal grid. (This is consistent with our isotropy discussion).

zero group velocity locations

zero phase velocity locations



So we have looked at isotropy, angular deficiency, mode counting and dispersion relations .... are we really any closer to picking a winner?

Summary comparison ....

	singularity	isotropy	mass/ velocity congruence	dispersion relation		
triangles	1	3	3	2		
squares	2	2	1	1		
hexagons	1	1	2	1		

So triangles, quads or hexagons ....
Does any one stand above the others?

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In terms its relatively weak grid singularities and isotropy, the hexagonal grid offers some clear advantages overall. The dispersion relation on the hexagonal grid is on-par with the quad grid, but only after moving beyond the obvious averaging schemes.

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Does any one stand above the others?

In terms its relatively weak grid singularities and isotropy, the hexagonal grid offers some clear advantages overall. The dispersion relation on the hexagonal grid is on-par with the quad grid, but only after moving beyond the obvious averaging schemes.

There are two reasons that often compel a choice other than hexagons. First, hexagons are not amenable to high-order basis methods (DG, spectral element, etc). Second, hexagons are not the natural choice for grid nesting, i.e. a regular hexagon can not be filled with smaller regular hexagons. Both triangles and quads do not suffer from either of these shortcomings.

Summary comparison ....

	singularity	isotropy	mass/ velocity congruence	dispersion relation	high-order methods	nesting
triangles	1	3	3	2	2	1
squares	2	2	1	1	1	1
hexagons	1	1	2	1	3	2

All are viable, meaning their respective deficiencies can be overcome or acceptably mitigated (at least for some subset of methods). It is often the case that the choice is made based on one aspect of grid.


Thank you